

# T U T O R I A L

## “Numerical Methods for the Solution of Elliptic Partial Differential Equations”

to the lecture

“Numerics of Elliptic Problems”

### **Tutorial 05**

Tuesday, 5 April 2011, Time: 10<sup>15</sup> – 11<sup>45</sup>, Room: SR / T 642.

**20** Let us consider the quadrature rule

$$\int_{\Delta} u(\xi) d\xi \approx u(\xi^*) |\Delta|,$$

with the unit triangle  $\Delta = \{\xi = (\xi_1, \xi_2) \in \mathbf{R}^2 : 0 < \xi_2 < 1 - \xi_1, 0 < \xi_1 < 1\}$  and the integration point  $\xi^* = (1/3, 1/3)$ . Show that there exists a positive constant  $c = \text{const.} > 0$  such that

$$\left| \int_{\Delta} u(\xi) d\xi - u(\xi^*) |\Delta| \right| \leq c \|u\|_{H^2(\Delta)} \quad \forall u \in H^2(\Delta).$$

**Hint:** In 2D ( $d = 2$ ),  $H^2(\Delta)$  is continuously (even compactly) embedded in  $C(\bar{\Delta})$ , i.e. there exists  $c_E = \text{const.} > 0 : \|u\|_{C(\bar{\Delta})} := \max_{\xi \in \Delta} |u(\xi)| \leq c_E \|u\|_{H^2(\Delta)}$ .

**21** Show that, for sufficiently smooth functions, e.g. for  $u, v \in H(\text{curl}) \cap [C^1(\bar{\Omega})]^3$ , the curl-IbyP-formula

$$\int_{\Omega} \text{curl}(u) \cdot v \, dx = \int_{\Omega} u \cdot \text{curl}(v) \, dx - \int_{\Gamma} (u \times n) \cdot v \, ds \quad (2.11)$$

is valid. **Hint:** Use the classical IbyP-formula for the proof of (2.11) !

**22** Let  $g \in H^{-1/2}(\Gamma) := (H^{1/2}(\Gamma))^*$  be a given flux. Show that there exist a unique weak (generalized) solution of the Neumann problem

$$-\Delta u + u = 0 \text{ in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial n} = g \text{ on } \Gamma = \partial\Omega \quad (2.12)$$

satisfying the apriori estimate

$$\|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 \leq c \|g\|_{H^{-1/2}(\Gamma)}^2.$$

with some positive constant  $c = ?$ .

**23\*** Show that the gradient  $q = \nabla u$  of the weak solution  $u$  of the Neumann problem (2.12) from Exercise 22 belongs to  $H(\text{div})$  and the weak divergence of  $q$  is equal to  $u$ , i.e.  $\text{div}(q) = u$  !

**24\*** Let  $\Omega_1, \dots, \Omega_m$  be a non-overlapping domain decomposition of  $\Omega$ , i.e.  $\bar{\Omega} = \cup \bar{\Omega}_i$ ,  $\Omega_i \cap \Omega_j = \emptyset$ ,  $i \neq j$ . Let  $q_i \in H(\text{div}, \Omega_i)$ ,  $i = 1, 2, \dots, m$ , such that

$$\gamma_{n_i, \Gamma_{ij}} q_i = \gamma_{n_i, \Gamma_{ij}} q_j \quad \forall \Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j : \text{meas}_{d-1} \Gamma_{ij} > 0.$$

Then the piecewise defined function

$$q := \{q|_{\Omega_i} = q_i, i = 1, 2, \dots, m\} \in H(\text{div}, \Omega) \text{ and } (\text{div}q)|_{\Omega_i} = \text{div}q_i,$$

for all  $i = 1, 2, \dots, m$ .