T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerics of Elliptic Problems"

Tutorial 04 Tuesday, 29 March 2011, Time: $10^{\underline{15}}$ – $11^{\underline{45}}$, Room: SR / T 642.

2 Tools from the Theory of Sobolev Spaces

15 Let us consider the function

$$u(x) = \begin{cases} 1+x, & -1 \le x \le 0 \\ 1-x, & 0 \le x \le -1 \end{cases},$$

Obviously, $u \in C(\overline{\Omega}) \subset L_p(\Omega) \subset L_{loc}(\Omega) \subset D'(\Omega)$, but $u \notin C^1(\overline{\Omega})$!
Compute

1. $u' = \partial^1 u \in$?
2. $u'' = \partial^2 u \in$?
3. $u''' = \partial^3 u \in$?

in the distributive sense !

16 Show that

$$\|g\|_{H^{1/2}(\Gamma)} = \inf_{u \in H^1(\Omega):\, \gamma_0 u = g} \|u\|_{H^1(\Omega)} \tag{2.10}$$

defines a norm in $H^{1/2}(\Gamma) := \gamma_0 H^1(\Omega)$ (check the norm axioms) ! The infimum in (2.10) is realized. Characterize the minimizer $u^* \in H^1(\Omega)$ as a unique solution of a variational problem !

17 Show that

$$\|u\|^*_{W^2_p(\Omega)} = \left(\int_\Gamma |u|^p ds + \int_\Gamma |\partial_n u|^p ds + |u|^p_{W^2_p(\Omega)} \right)^{1/p}$$

defines a new norm in $W^2_p(\Omega)$ that is equivalent to the standard norm

$$\|u\|_{W^2_p(\Omega)} = \left(\sum_{|\alpha| \le 2} \int_\Omega |\partial^\alpha u|^p dx \right)^{1/p} = \left(\int_\Omega |u|^p dx + \int_\Omega |\nabla u|^p dx + |u|^p_{W^2_p(\Omega)} \right)^{1/p},$$

where $\partial_n u(x) = \frac{\partial u}{\partial n}(x) = (\nabla u(x), n(x)) = \nabla u(x)^T n(x) = \nabla u(x) \bullet n(x)$, and $|u|_{W^2_p(\Omega)} = (\sum_{|\alpha|=2} \int_\Omega |\partial^\alpha u|^p dx)^{1/p}$ denotes the standard semi-norm in $W^2_p(\Omega)$.

18 Show that there exists a positive constant $c_F = const > 0$ such that

$$\int_\Omega (u(x))^2 dx \leq c_F^2 \int_\Omega |\nabla u(x)|^2 dx \quad \forall u \in V_0 = H_0^1(\Omega)$$

with

$$c_F = \frac{1}{\sqrt{2}} \min_{i=1,\ldots,d} (b_i - a_i),$$

where $\Omega \subset \Pi := \{x = (x_1, \ldots, x_d) \in \mathbf{R}^d : a_i < x_i < b_i,\ i = 1, \ldots, d\}$.

19 Show that there exists a positive constant $c = const > 0$ such that

$$\inf_{q \in \mathbf{R}} \|u - q\|_{L_p(\Omega)} \leq c \, \|\nabla u\|_{L_p(\Omega)} \quad \forall u \in W_p^1(\Omega).$$