

# T U T O R I A L

## “Numerical Methods for the Solution of Elliptic Partial Differential Equations”

to the lecture

“Numerics of Elliptic Problems”

### **Tutorial 02**

Tuesday, 15 March 2011, Time: 10<sup>15</sup> – 11<sup>45</sup>, Room: SR / T 642.

**06\*** Derive the variational formulation of the Dirichlet problem for the Helmholtz equation

$$(**) \begin{cases} -\Delta u(x) - \omega^2 u = f(x), x \in \Omega \\ u(x) = 0, x \in \Gamma = \partial\Omega \end{cases} .$$

Then discuss the problem of the existence and the uniqueness of the generalized solution of (\*\*)! Here  $\omega^2$  is a given positive constant.

## 1.2 The linear elasticity problem

**07** Show that, for the BVP of the first type ( $\Gamma_1 = \Gamma$ ) and for the mixed BVP ( $\text{meas}_2(\Gamma_1) > 0$  and  $\text{meas}_2(\Gamma_2) > 0$ ) of the linear elasticity, the following statements hold:

1.  $a(\cdot, \cdot)$  is symmetric, i.e.,  $a(u, v) = a(v, u) \quad \forall u, v \in V$ ,
2.  $a(\cdot, \cdot)$  is nonnegative, i.e.,  $a(v, v) \geq 0 \quad \forall v \in V$ ,
3.  $a(\cdot, \cdot)$  is positive on  $V_0 := \{v \in V = [H^1(\Omega)]^3 : v = 0 \text{ on } \Gamma_1\}$ , if  $\text{meas}_2(\Gamma_1) > 0$ , i.e.,  $a(v, v) > 0 \quad \forall v \in V_0 : v \neq 0$ .

The equivalence of VF (9)<sub>VF</sub> and MP (9)<sub>MP</sub> given in the Lectures then follows from Section. 1.1. of the Lectures.

**08** Show that, for the first type ( $\Gamma_1 = \Gamma$ ) BVP of the 3D linear elasticity in the case of an isotrop and homogeneous material, the assumptions of the Lax-Milgram Theorem are fulfilled, i.e. provide constants  $\mu_1$  and  $\mu_2$  such that:

- 1)  $F \in V_0^*$ ,
- 2a)  $\exists \mu_1 = \text{const} > 0 : a(v, v) \geq \mu_1 \|v\|_{H^1(\Omega)}^2 \quad \forall v \in V_0$ ,
- 2b)  $\exists \mu_2 = \text{const} > 0 : |a(u, v)| \leq \mu_2 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \quad \forall u, v \in V_0$ .

○ Hint: to the proof of  $V_0$ -ellipticity:

$$+ a(v, v) \geq 2\mu \int_{\Omega} \sum_{i,j=1}^3 (\varepsilon_{ij}(v))^2 dx,$$

- + Korn's inequality for the BVP of the first type:  $V_0 = [H_0^1(\Omega)]^3$ ,  
where  $H_0^1(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ auf } \Gamma\}$  (Prove this inequality ! Use integration by parts !),
- + FRIEDRICHS-inequality.

**09** Provide the weak form of the iterative method (3) from Section 1.1 of the Lectures

$$u_{n+1} = u_n - \rho(JAu_n - JF) \text{ in } V_0 = (H_0^1(\Omega))^3, \quad (1.4)$$

with  $n = 0, 1, 2, \dots$ , and given  $u_0 \in V_0$ , for the first BVP of the linear elasticity in case of 3D homogeneous and isotrop material, i.e., derive the weak form (variation formulation) for the calculation of  $u_{n+1} \in V_0$  ! Discuss two cases, in which the norm on  $V_0$  is defined as

$$\|u\|_{V_0}^2 := \int_{\Omega} |\nabla u|^2 dx \quad \forall u \in V_0, \quad (1.5)$$

or as

$$\|u\|_{V_0}^2 := \int_{\Omega} (|\nabla u|^2 + |u|^2) dx \quad \forall u \in V_0. \quad (1.6)$$

**10\*** Let us consider the variational formulation:

$$\text{Find } u \in V_g = V_0 : \quad a(u, v) = \langle F, v \rangle \quad \forall v \in V_0 \quad (1.7)$$

of a plane linear elasticity problem in  $\Omega = (0, 1) \times (0, 1)$ , where

$$\begin{aligned} V_0 = & \{ u = (u_1, u_2) \in V = [H^1(\Omega)]^2 : \\ & u_1 = 0 \text{ on } \Gamma_2 = \{0\} \times [0, 1] \\ & u_2 = 0 \text{ on } [0, 1] \times \{1\} \}, \\ a(u, v) = & \int_{\Omega} \sum_{i,j,k,l=1}^2 D_{ijkl} \varepsilon_{ij}(u) \varepsilon_{kl}(v) dx = \int_{\Omega} \sum_{k,l=1}^2 \sigma_{kl}(u) \varepsilon_{kl}(v) dx, \\ \langle F, v \rangle = & \int_{\Omega} \sum_{i=1}^2 f_i v_i dx + \int_{\Gamma_1} ? ds + \int_{\Gamma_2} ? ds. \end{aligned}$$

Impose the right natural BCs ! Give the classical formulation of (1.7) !.