

T U T O R I A L

“Numerical Methods for the Solution of Elliptic Partial Differential Equations”

to the lecture

“Numerics of Elliptic Problems”

Tutorial 01

Thursday, 10 March 2011, Time: 10¹⁵ – 11⁴⁵, Room: SR / T 111.

1 Variational formulation of multi-dimensional elliptic Boundary Value Problems (BVP)

1.1 Scalar Second-order Elliptic BVP

○ In the lecture (Section 1.2.1) we discussed the BVP (a classical formulation)

$$\begin{aligned}
 & \text{Search } u \in X := C^2(\Omega) \cap C^1(\Omega \cup \Gamma_2 \cup \Gamma_3) \cap C(\Omega \cup \Gamma_1) : & (1.1) \\
 & - \sum_{i,j=1}^d \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) + \sum_{i=1}^d b_i(x) \frac{\partial u}{\partial x_i} + c(x)u(x) = f(x), x \in \Omega \\
 & +\text{BC: } \bullet \quad u(x) = g_1(x), x \in \Gamma_1 \\
 & \bullet \quad \frac{\partial u}{\partial N} := \sum_{i,j=1}^d a_{ij}(x) \frac{\partial u(x)}{\partial x_j} n_i(x) = g_2(x), x \in \Gamma_2 \\
 & \bullet \quad \frac{\partial u}{\partial N} + \alpha(x)u(x) = \underbrace{g_3(x)}_{\alpha(x)u_A(x)}, x \in \Gamma_3
 \end{aligned}$$

and derived the variational formulation

$$\begin{aligned}
 & \text{Search } u \in V_g : a(u, v) = \langle F, v \rangle \quad \forall v \in V_0 & (1.2) \\
 & \text{with} \\
 & a(u, v) := \int_{\Omega} \left(\sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \sum_{i=1}^d b_i \frac{\partial u}{\partial x_i} v + cuv \right) dx + \int_{\Gamma_3} \alpha uv ds, \\
 & \langle F, v \rangle := \int_{\Omega} f v dx + \int_{\Gamma_2} g_2 v ds + \int_{\Gamma_3} g_3 v ds \\
 & V_g := \{v \in V = W_2^1(\Omega) : v = g_1 \text{ on } \Gamma_1\} \\
 & V_0 := \{v \in V : v = 0 \text{ on } \Gamma_1\}
 \end{aligned}$$

under the assumptions

$$\left. \begin{array}{l}
 1) \quad a_{ij}, b_i, c \in L_\infty(\Omega), \alpha \in L_\infty(\Gamma_3) \\
 2) \quad f \in L_2(\Omega), g_i \in L_2(\Gamma_i), i = 2, 3 \\
 3) \quad g_1 \in H^{\frac{1}{2}}(\Gamma_1), \text{ i.e., } \exists \tilde{g}_1 \in H^1(\Omega) : \tilde{g}_1|_{\Gamma_1} = g_1. \\
 4) \quad \Omega \subset \mathbf{R}^d (\text{bounded}) : \Gamma = \partial\Omega \in C^{0,1} \text{ (Lipschitz-continuous boundary)} \\
 5) \quad \text{uniform ellipticity:} \\
 \left. \begin{array}{l}
 \sum_{i,j=1}^d a_{ij}(x)\xi_i; \xi_j \geq \bar{\mu}_1 |\xi|^2 \quad \forall \xi \in \mathbf{R}^d \\
 a_{ij}(x) = a_{ji}(x) \quad \forall i, j = \overline{1, d}
 \end{array} \right\} \forall \text{ a.e. } x \in \Omega.
 \end{array} \right\} \quad (1.3)$$

01 Formulate the classical assumptions on $\{a_{ij}, b_i, c, \alpha, f, g_i, \Omega \text{ resp. } \partial\Omega\}$ for (1.1) !

02 Show that, for sufficiently smooth data, a the generalized solution $u \in V_g \cap X \cap H^2(\Omega)$ of the Neumann-Problem (2) is also a classical solution, i.e. a solution of (1) !

$$(1) \quad \left\{ \begin{array}{l}
 \text{Find } u \in X = C^2(\Omega) \cap C^1(\bar{\Omega}) : \\
 -\Delta u(x) + u(x) = f(x), x \in \Omega \subset \mathbf{R}^d \text{ (bounded)}, \\
 \frac{\partial u}{\partial n}(x) = g(x), x \in \Gamma = \partial\Omega
 \end{array} \right.$$

? \Downarrow \Uparrow ?

$$(2) \quad \left\{ \begin{array}{l}
 \text{Find } u \in V_g = H^1(\Omega) : \\
 \underbrace{\int_{\Omega} (\nabla^T u \nabla v + uv) dx}_{=a(u,v)} = \underbrace{\int_{\Omega} f v dx + \int_{\Gamma} g v ds}_{= \langle F, v \rangle} \quad \forall v \in V_0 = H^1(\Omega) = W_2^1(\Omega)
 \end{array} \right.$$

03 Show that the assumptions of the Lax-Milgram-Theorem are satisfied for the variational problem (1.2) under the assumptions (1.3) and the additional assumptions $b_i = 0, c = 0, \alpha(x) \geq \underline{\alpha} = \text{const} > 0 \quad \forall \text{ a.e. } x \in \Gamma_3$ with $\text{meas}_{d-1}(\Gamma_i) > 0, i = 1, 2, 3$!

04 In addition to the assumption (1.3) let us assume that $c(x) \geq \underline{c} = \text{const} > 0 \quad \forall \text{ a.e. } x \in \Omega, \Gamma_2 = \Gamma_3 = \emptyset, g_1 = 0$, and $b_i \neq 0$. Provide conditions for the coefficients $b_i(x)$ respectively for the vector function $b(x) = (b_1(x), \dots, b_d(x))^T$ such that the the assumptions of Lax-Milgram's Theorem are fulfilled !

○ Hint: For estimating the convection term

$$\sum_{i=1}^m \int_{\Omega} b_i(x) \frac{\partial u}{\partial x_i}(x) v(x) dx = \int_{\Omega} (b(x), \nabla u(x)) v(x) dx,$$

make use of the identity

$$\int_{\Omega} (b, \nabla u) v dx = - \int_{\Omega} (b, \nabla v) u dx - \int_{\Omega} u v \text{div}(b) dx + \int_{\Gamma} (b, n) u v ds$$

that can easily be proved by partial integration.

05* Derive the variational formulation of the pure Neumann problem for the Poisson equation

$$(*) \begin{cases} -\Delta u(x) = f(x), x \in \Omega \\ \frac{\partial u}{\partial \vec{n}}(x) = 0, x \in \Gamma = \partial\Omega \end{cases},$$

and discuss the question of the existence and the uniqueness of a generalized solution of (*) !

○ Hint:

Obviously, $u(x)+c$ with an arbitrary constant $c \in \mathbf{R}^1$ solves (*), if u is the solution of the BVP (*). There are the following ways to analyze the existence of a generalized solution:

- 1) Set up the variational formulation in $V = H^1(\Omega)$ and apply FREDHOLM-Theory !
- 2) Set up the variational formulation in factor-space $V = H^1(\Omega)|_{\ker}$ with $\ker = \{c : c \in \mathbf{R}^1\} = \mathbf{R}^1$ and apply LAX-MILGRAM-Theorem !