2.7. Sobolev's Embedding Theorem

■ **Def. 2.23:**

X, Y – B-spaces: $X \subset Y$ (as sets).
The embedding operator $E: X \to Y$ assigne every element $u \in X$ the same element $u \in Y$. (after identification!)
The embedding is called continuous resp. compact iff the embedding operator E is continuous resp. compact.
Notation: $X \subset Y$ – continuous embedding,
$X \hookrightarrow Y$ – compact embedding.

■ **Theorem 2.24:** (Embedding for $\mathring{W}^1_p(\Omega)$)

Ass.: $\Omega \subset \mathbb{R}^d$ * ($\partial\Omega \in C^{0,1}$ is not required!)
$1 < p, q < \infty$

St.: 1. $\mathring{W}^1_p \hookrightarrow C(\bar{\Omega})$ if $p > d$,
i.e. for every function (equivalence class of functions) $u \in \mathring{W}^1_p(\Omega)$ there exists an equivalent function $u \in C(\bar{\Omega})$, and the embedding operator
$E \in L(\mathring{W}^1_2, C(\bar{\Omega}))$ and compact.

2. Let now $p \le d$. Then
a) $\mathring{W}^1_p(\Omega) \subset L_q(\Omega)$ if $q \le q_* := \frac{pd}{d-p} \wedge q < \infty$.
b) $\mathring{W}^1_p(\Omega) \hookrightarrow L_q(\Omega)$ if $q < q_*$.

3. Let $p \le d$ and $\Omega_m = \Omega \cap \mathcal{H}_m$: $\operatorname{meas}_{\mathbb{R}^m}(\Omega_m) > 0$, where $\mathcal{H}_m$ is an m-dim. Hyperplane: $d-p \le m$, $m \le d$, $m \ge 1$.
a) $\mathring{W}^1_p(\Omega) \subset L_q(\Omega_m)$ if $q \le q_{**} := \frac{pm}{d-p} \wedge q < \infty$.
b) $\mathring{W}^1_p(\Omega) \hookrightarrow L_q(\Omega_m)$ if $q < q_{**}$.