01 Show (using the trace theorems from the lecture) that

$$|||v|||_{H^{1/2}(\Gamma)} \;:=\; \inf_{\widetilde{v}\in H^1(\Omega):\ \widetilde{v}_{|\Gamma}=v} \|\widetilde{v}\|_{H^1(\Omega)}$$

is an equivalent norm to $\|\cdot\|_{H^{1/2}(\Gamma)}$.

02 Let Ω be a bounded Lipschitz domain. Assume that you know that there exists a parameter $\widetilde{C}_P(\Omega)$ such that

$$\|u\|^2_{L^2(\Omega)} \;\le\; \widetilde{C}_P(\Omega)\,|u|^2_{H^1(\Omega)} \qquad \forall u\in H^1(\Omega),\ \overline{u}^\Omega = 0 \tag{1.1}$$

(e.g., by an indirect proof using Rellich's theorem – the compact embedding of $L^2(\Omega)$ in $H^1(\Omega)$). Show:

(i) $\|u-\overline{u}^\Omega\|^2_{L^2(\Omega)} \;\le\; \widetilde{C}_P(\Omega)\,|u|^2_{H^1(\Omega)} \qquad \forall u\in H^1(\Omega)$

(ii) $\|u\|^2_{L^2(\Omega)} \;\le\; 2\,\widetilde{C}_P(\Omega)\,|u|^2_{H^1(\Omega)} + \dfrac{2}{|\Omega|}\Big(\displaystyle\int_\Omega u\,dx\Big)^2 \qquad \forall u\in H^1(\Omega)$

(iii) $\inf_{c\in\mathbb{R}} \|u-c\|^2_{L^2(\Omega)} \;=\; \|u-\overline{u}^\Omega\|^2_{L^2(\Omega)} \qquad \forall u\in H^1(\Omega)$

Hint: the left hand side is a real-valued quadratic function with respect to c.

(iv) BONUS EXAMPLE: Show that there exists a parameter $C_P(\Omega)>0$ that is independent of $\operatorname{diam}(\Omega)$ such that

$$\|u-\overline{u}^\Omega\|^2_{L^2(\Omega)} \;\le\; C_P(\Omega)\,\operatorname{diam}(\Omega)^2\,|u|^2_{H^1(\Omega)} \qquad \forall u\in H^1(\Omega).$$

Hint: transform Ω to a domain with diameter 1 and use (1.1) there.

03 Let $\mathcal{T}^h(\Omega)$ be a regular triangulation of Ω and let $V_D^h(\Omega)$ be the space of continuous piecewise linear finite element functions that satisfy the homogeneous Dirichlet boundary conditions on Γ_D. Recall that to each node x_i in $\Omega\setminus\Gamma_D$, we associate a nodal basis function $\varphi_i\in V_D^h(\Omega)$. Let the functionals $\psi_i\in V_D^h(\Omega)^*$ be defined as

$$\langle\psi_i,\,v\rangle := v(x_i) \qquad \text{for } i=1,\dots,n_h$$

and note that $\psi_i(\varphi_j)=\delta_{ij}$. Show that

(i) $a(v_h,\,w_h) \;=\; (K_h\underline{v}_h,\,\underline{w}_h)_{\ell^2} \qquad \forall v_h,\,w_h\in V_D^h(\Omega)$

(ii) $\langle F,\,v_h\rangle \;=\; (\underline{f}_h,\,\underline{v}_h)_{\ell^2} \qquad \forall v_h\in V_D^h(\Omega)$

(iii) $\forall F\in V_D^h(\Omega)^*:\ F=\displaystyle\sum_{i=1}^{n_h}\langle F,\,\varphi_i\rangle\,\psi_i$

where $(\cdot,\cdot)_{\ell^2}$ denotes the Euclidean inner product in $\mathbb{R}^{n_h}$, and the entries of the vectors $\underline{v}_h$, $\underline{w}_h$ are the basis coefficients of v_h, w_h, respectively.