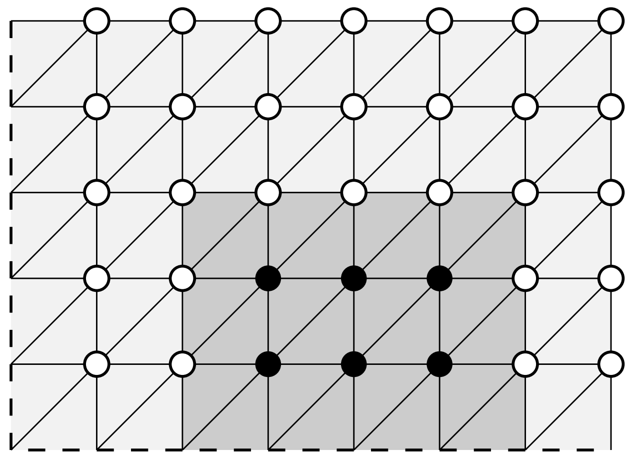
“bilinear” world	“operator” world
bilinear form $a : V \times V \to \mathbb{R}$	linear operator $A : V \to V^*$
$a(w,\, v) \;=\; \langle A\, w,\, v \rangle$	
$a(\cdot,\cdot)$ symmetric	A self-adjoint
$a(\cdot,\cdot)$ coercive	A positive definite



- - Dirichlet boundary Γ_D

$\circ$ nodes in $\overline{\Omega} \setminus (\Gamma_D \cup \Omega_i)$

$\bullet$ nodes in (the interior of) Ω_i

Additive Schwarz method – overview

On Hilbert space $\boldsymbol{V}$:

$$\text{bilinear form } \boldsymbol{a}(\cdot,\cdot) \quad \longleftrightarrow \quad \text{operator } \boldsymbol{A}$$

On “sub”spaces $\{\boldsymbol{V_i}\}_{i=1}^{N}$:

prolongation operators	$\boldsymbol{R}_i^\top : V_i \to V$	(linear and injective)
restriction operators	$\boldsymbol{R_i} : V^* \to V_i^*$	
local bilinear operators	$\boldsymbol{A_i} := \boldsymbol{R_i}\,\boldsymbol{A}\,\boldsymbol{R}_i^\top \quad \longleftrightarrow \quad \boldsymbol{a_i}(\cdot,\cdot)$	

Schwarz projection operators:

$$\boldsymbol{P_i} \;=\; \boldsymbol{R_i}\, \underbrace{\boldsymbol{A}_i^{-1}\,\boldsymbol{R}_i^\top\,\boldsymbol{A}}_{=\,\widetilde{\boldsymbol{P}}_i} \qquad\qquad \boldsymbol{a_i}(\widetilde{\boldsymbol{P}}_i\,\boldsymbol{v},\,\boldsymbol{w_i}) \;=\; \boldsymbol{a}(\boldsymbol{v},\,\boldsymbol{R}_i^\top\boldsymbol{w_i})$$

- $\boldsymbol{P_i}$ is **self-adjoint** and **positive definite** w.r.t. $\boldsymbol{a}(\cdot,\cdot)$
- $\boldsymbol{P}_i^2 = \boldsymbol{P_i}$
- $\|\boldsymbol{P_i}\|_a \leq 1$

Additive Schwarz operator / preconditioner:

$$\boldsymbol{P}_{\text{ad}} := \sum_{i=1}^{N} \boldsymbol{P_i} \qquad\qquad \boldsymbol{C}_{\text{ad}}^{-1} := \sum_{i=1}^{N} \boldsymbol{R}_i^\top\,\boldsymbol{A}_i^{-1}\,\boldsymbol{R_i} \qquad\qquad \boldsymbol{P}_{\text{ad}} \;=\; \boldsymbol{C}_{\text{ad}}^{-1}\,\boldsymbol{A}$$

$$\boldsymbol{a}(\boldsymbol{P}_{\text{ad}}\,\boldsymbol{u},\,\boldsymbol{v}) \;=\; \sum_{i=1}^{N} \boldsymbol{a_i}(\widetilde{\boldsymbol{P}}_i\,\boldsymbol{u},\,\widetilde{\boldsymbol{P}}_i\,\boldsymbol{v}) \;=\; \boldsymbol{a}(\boldsymbol{u},\,\boldsymbol{P}_{\text{ad}}\,\boldsymbol{v}) \tag{2.7}$$

Algorithm 1: Application of the multiplicative Schwarz preconditioner

$\boldsymbol{r} \in \boldsymbol{V}^*$ given
$\boldsymbol{x} = \boldsymbol{0}$
for $i = 1, \ldots, N$ **do**
  $\boldsymbol{x} = \boldsymbol{x} + \boldsymbol{R}_i^\top \boldsymbol{A}_i^{-1} \boldsymbol{R}_i \, (\boldsymbol{r} - \boldsymbol{A}\, \boldsymbol{x})$
end
return $\boldsymbol{x}$ $(= \boldsymbol{C}_{\text{mu}}^{-1} \boldsymbol{r})$

Algorithm 2: Projected PCG involving hybrid Schwarz preconditioner

$\boldsymbol{u}^{(0)} = \boldsymbol{R}_0^\top \boldsymbol{A}_0^{-1} \boldsymbol{R}_0 \, \boldsymbol{f} + \widetilde{\boldsymbol{u}}^{(0)}$, $\widetilde{u}^{(0)} \in \text{range}(I - P_0)$
$\boldsymbol{r}^{(0)} = \boldsymbol{f} - \boldsymbol{A}\, \boldsymbol{u}^{(0)}$
$\boldsymbol{k} = \boldsymbol{0}$
repeat
  $\boldsymbol{y}^{(k)} = (\boldsymbol{I} - \boldsymbol{P}_0^\top)\, \boldsymbol{r}^{(k)}$ **(project)**
  $\boldsymbol{z}^{(k)} = \sum_{i=1}^{N} \boldsymbol{R}_i^\top \boldsymbol{A}_i^{-1} \boldsymbol{R}_i \, \boldsymbol{y}^{(k)}$ **(precondition)**
  $\boldsymbol{s}^{(k)} = (\boldsymbol{I} - \boldsymbol{P}_0)\, \boldsymbol{z}^{(k)}$ **(project)**
  if $k = 0$ **then**
    $\boldsymbol{p}^{(k)} = \boldsymbol{s}^{(k)}$
  else
    $\boldsymbol{\beta}_{k-1} = \frac{(\boldsymbol{r}^{(k)}, \boldsymbol{s}^{(k)})_{\ell^2}}{(\boldsymbol{r}^{(k-1)}, \boldsymbol{s}^{(k-1)})_{\ell^2}}$
    $\boldsymbol{p}^{(k)} = \boldsymbol{s}^{(k)} + \boldsymbol{\beta}_{k-1}\, \boldsymbol{p}^{(k-1)}$
  end
  $\boldsymbol{\alpha}_k = \frac{(\boldsymbol{r}^{(k)}, \boldsymbol{s}^{(k)})_{\ell^2}}{(\boldsymbol{A}\, \boldsymbol{p}^{(k)}, \boldsymbol{p}^{(k)})_{\ell^2}}$
  $\boldsymbol{u}^{(k+1)} = \boldsymbol{u}^{(k)} + \boldsymbol{\alpha}_k \, \boldsymbol{p}^{(k)}$
  $\boldsymbol{r}^{(k+1)} = \boldsymbol{r}^{(k)} - \boldsymbol{\alpha}_k \, \boldsymbol{A}\, \boldsymbol{p}^{(k)}$
  $\boldsymbol{k} = \boldsymbol{k} + \boldsymbol{1}$
until *stopping criterion fulfilled*