$$(*) = \int_{x_{i-1}}^{x_i} \Big[\frac{1}{h_i} \int_{x_{i-1}}^{x_i} 1 \cdot \int_{\xi}^{x} z''(\eta)\, d\eta \, d\xi \Big]^2 dx$$

Cauchy

$$\le \int_{x_{i-1}}^{x_i} \Big\{ \frac{1}{h_i^2} \int_{x_{i-1}}^{x_i} 1^2 d\xi \int_{x_{i-1}}^{x_i} \Big(\int_{\xi}^{x} z''(\eta)\, d\eta \Big)^2 d\xi \Big\} dx$$

$$= \int_{x_{i-1}}^{x_i} \Big\{ \frac{1}{h_i^2} \underbrace{(x_i - x_{i-1})}_{= h_i} \int_{x_{i-1}}^{x_i} \underbrace{\Big(\int_{\xi}^{x} 1 \cdot z''(\eta)\, d\eta \Big)^2}_{\overset{\text{Cauchy}}{\le} \left| \int_{\xi}^{x} 1^2 d\eta \right| \cdot \left| \int_{\xi}^{x} (z''(\eta))^2 d\eta \right| \;\le\; \underbrace{\int_{x_{i-1}}^{x_i} 1^2 d\eta}_{= x_i - x_{i-1} = h_i} \cdot \int_{x_{i-1}}^{x_i} (z''(\eta))^2 d\eta} d\xi \Big\} dx$$

$$\le \int_{x_{i-1}}^{x_i} \frac{1}{h_i^2} h_i \int_{x_{i-1}}^{x_i} h_i \int_{x_{i-1}}^{x_i} (z''(\eta))^2 d\eta \, d\xi \, dx$$

$$= \int_{x_{i-1}}^{x_i} dx \int_{x_{i-1}}^{x_i} d\xi \int_{x_{i-1}}^{x_i} (z''(\eta))^2 d\eta =$$

$$= h_i \cdot h_i \int_{x_{i-1}}^{x_i} (u''(x) - \underbrace{\tilde{u}_h''(x)}_{=0})^2 dx = h_i^2 \int_{x_{i-1}}^{x_i} (u''(x))^2 dx$$

<u>Resultat:</u>

$$\boxed{\int_{x_{i-1}}^{x_i} |z'(x)|^2 dx \le h_i^2 \int_{x_{i-1}}^{x_i} (u''(x))^2 dx}$$

Damit haben wir gezeigt:

$$\| u - \tilde{u}_h \|_1^2 \le (1 + c_F^2) | u - \tilde{u}_h |_1^2 = (1 + c_F^2) | z |_1^2$$

$$= (1 + c_F^2) \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} | z'(x) |^2 dx \le$$

$$\le (1 + c_F^2) \sum_{i=1}^{n} h_i^2 \int_{x_{i-1}}^{x_i} (u''(x))^2 dx$$

$h = \max h_i$
$u \in H^2(a,b)$

$$\le (1 + c_F^2)\, h^2 \, \| u'' \|_{L_2(a,b)}.$$

q.e.d.