T U T O R I A L

"Computational Mechanics"

to the lecture

"Numerical Methods in Continuum Mechanics 1"

Tutorial 10/11 Wedneseday, Jun 09, 2010
(Time : $8^{\underline{30}}$ – $10^{\underline{00}}$ Room : SR T 111)

36 Show, that the preconditioned Uzawa Algorithm (see formula (51) in Chapter 2 of the Lectures) is equivalent to the classical Uzawa Algorithm (see formula (48) in Chapter 2 of the Lectures) applied to the preconditioned system

$$\begin{aligned} A\underline{u} + B^T D^{-1/2}\underline{\mu} &= \underline{f}\,, \\ D^{-1/2}B\underline{u} - D^{-1/2}CD^{-1/2}\underline{\mu} &= D^{-1/2}\underline{g}\,. \end{aligned}$$

37 Let us consider the problem $S\underline{X} = \underline{F}$, where

$$S = \begin{pmatrix} (A-A_0)A_0^{-1}A & (A-A_0)A_0^{-1}B^T \\ BA_0^{-1}(A-A_0) & BA_0^{-1}B^T + C \end{pmatrix}, \; \underline{X} = \begin{pmatrix} \underline{u} \\ \underline{\lambda} \end{pmatrix}, \; \underline{F} = \begin{pmatrix} (A-A_0)A_0^{-1}\underline{f} \\ BA_0^{-1}\underline{f} - \underline{g} \end{pmatrix}.$$

(see formula (61) in Chapter 2 of the Lectures). Write down in detail (for $\underline{u}_k$ and $\underline{\lambda}_k$) the preconditioned Richardson-method

$$\bar{S}\,\frac{\underline{X}_{k+1} - \underline{X}_k}{\tau} + S\underline{X}_k = \underline{F}\,, \tag{4.62}$$

with

$$\bar{S} = \begin{pmatrix} A - A_0 & 0 \\ 0 & D \end{pmatrix},$$

where D is a good preconditioner for the Schur complement $BA^{-1}B^T + C$, c.f. spectral equivalence inequalities (71) from Chapter 2 of the Lectures. Which error estimates do you know ?

38 Describe the relation of the preconditioned Richardson Method (4.62) with $A_0 = \gamma G$ and the Arrow-Hurwicz Algorithm (see formula (54) in Chapter 2 of the Lectures).

39 Provide the detailed (i.e. for the iterates $\underline{u}_k$ and $\underline{\lambda}_k$) algorithm of the so-called Bramble-Pasciak-CG which is nothing else but the preconditioned (with $\bar{S}$) CG for solving system $S\underline{X} = \underline{F}$ (this is system (61) from Chapter 2 of the Lectures), and provide the corresponding iteration error estimate !

Hint: The preconditioning equation $\bar{S}\,\underline{W} = \underline{R}$ has the residual (defect)

$$\underline{R} = \underline{F} - S\underline{X} = \begin{pmatrix} (A-A_0)A_0^{-1}(\underline{f} - A\underline{u} - B^T\underline{\lambda}) \\ B[A_0^{-1}(\underline{f} - A\underline{u} - B^T\underline{\lambda}) + u] - C\underline{\lambda} - \underline{g} \end{pmatrix}. \tag{4.63}$$

as the right-hand side showing that the inversion of $A - A_0$ is not necessary (see also Exercise 2.24 from Chapter 2 of the Lecture) !

$\boxed{40^*}$ Consider the discrete mixed variational problem: Find $(u_h, \lambda_h) \in X_h \times \Lambda_h$ such that

$$\begin{aligned} a(u_h, v_h) + b(v_h, \lambda_h) &= \langle F\,,\, v_h\rangle \quad \forall v_h \in X_h\,, &(4.64)\\ b(u_h, \mu_h) &= \langle G\,,\, \mu_h\rangle \quad \forall \mu_h \in \Lambda_h\,. &(4.65)\end{aligned}$$

Let $\{\phi^{(i)}\}$ be a basis for X_h and $\{\varphi^{(k)}\}$ be a basis for Λ_h. Then, the discrete solutions u_h and λ_h can be represented by

$$u_h := \sum_i u_i \phi^{(i)}\,,\ \lambda_h := \sum_k \lambda_k \varphi^{(k)}\,,$$

and the problem (4.64)–(4.65) can equivalently written as: Find $(\underline{u}, \underline{\lambda})$ such that

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{g} \end{pmatrix}\,, \tag{4.66}$$

where

$$A = \big(a(\phi^{(j)}, \phi^{(i)})\big)_{ij}\,,\quad B = \big(b(\phi^{(j)}, \varphi^{(k)})\big)_{kj}\,,\quad \underline{f} = \big(\langle F\,,\, \phi^{(i)}\rangle\big)_i\,,\quad \underline{g} = \big(\langle G\,,\, \varphi^{(k)}\rangle\big)_k\,.$$

Show, that under the assumptions

1. the bilinearform a is symmetric, elliptic and bounded in the whole space X (e. g. Stokes),
2. the bilinearform b is bounded, i. e.,

   $$|b(v, \mu)| \le \beta_2\, \|v\|_X \|\mu\|_\Lambda\,,$$

3. the discrete inf-sup condition is satisfied, i. e.,

   $$\inf_{0\ne\mu_h\in\Lambda_h} \sup_{0\ne v_h \in X_h} \frac{b(v_h, \mu_h)}{\|v_h\|_X \|\mu_h\|_\Lambda} \ge \tilde{\beta}_1 > 0\,,$$

   where $\tilde{\beta}_1$ is independent of h,

the matrix $M = \big((\varphi^{(l)}\,,\, \varphi^{(k)})_\Lambda\big)_{kl}$ is a preconditioner for the Schur-complement $S = BA^{-1}B^T$, i. e., there exist positive constants $\underline{\gamma}$ and $\overline{\gamma}$ such that

$$\underline{\gamma} M \le S \le \overline{\gamma} M\,.$$

Hint: Since a is bounded and elliptic on the whole space, we can define $\|\cdot\|_X := a(\cdot,\cdot)^{1/2}$. Show, that

$$\big(BA^{-1}B^T \underline{\mu}\,,\, \underline{\mu}\big)_{l_2} = \sup_{0\ne v_h\in X_h} \frac{b(v_h, \mu_h)^2}{\|v_h\|_V^2}\,.$$

Then, use the discrete inf-sup condition and the boundedness for $b(\cdot,\cdot)$.