TUTORIAL

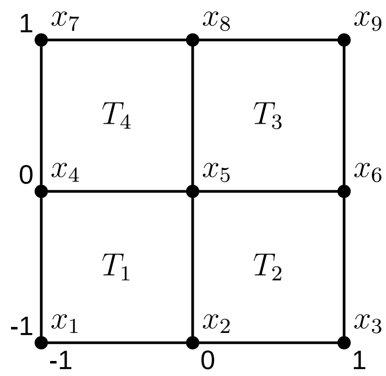
"Computational Mechanics"

to the lecture

"Numerical Methods in Continuum Mechanics 1"

Tutorial 08 Friday, May 21, 2010 (Time : $10^{\underline{15}}$ – $11^{\underline{00}}$, Room : HS 14)

28 Consider the macro element $M = (-1,1) \times (-1,1)$, consisting of four squares $T_1, \ldots, T_4$ and nine gridpoints $x_1, \ldots, x_9$:



The edges of M are denoted $S_1 = [x_3, x_9]$, $S_2 = [x_9, x_7]$, $S_3 = [x_7, x_1]$, $S_4 = [x_1, x_3]$, the union over all squares in $\{T_1, \ldots, T_4\}$ that contain the grid point x_i is denoted Δ_i, and the area of Δ_i is denoted $|\Delta_i|$. Show, that for each function in $C(\overline{M})$ there exists a unique Function $v_h = \Pi_h v \in C(\overline{M})$ which is bilinear (=quadrilinear) on each piece in $\{T_1, \ldots, T_4\}$, and satisfies

$$\forall i \in \{1,3,5,7,9\} : \; v_h(x_i) = \frac{1}{|\Delta_i|} \int_{\Delta_i} v \, \mathrm{d}x \, , \; \forall j \in \{1,2,3,4\} : \; \int_{S_j} v_h \, \mathrm{d}s = \int_{S_j} v \, \mathrm{d}s \, .$$

For $i \in \{1, \ldots, 9\}$, let $\varphi^{(i)}$ be the piecewise bilinear nodal ansatz functions which satisfies the relation $\varphi^{(i)}(x_j) = \delta_{ij}$. The function v_h, as defined above, can be written as

$$v_h(x) = \sum_{i=1}^{9} \alpha_i \, \varphi^{(i)}(x) \, . \tag{4.57}$$

Calculate the coefficients α_i explicitely!

29 Consider the assumptions and definitions in Example 28. Show, that there exists a constant $c_F > 0$, such that for all $v \in C^1(\overline{M})$ there holds $\|\Pi_h v\|_{H^1(M)} \le c_F \|v\|_{H^1(M)}$.
Hint: Use the representation (4.57). The coefficients α_i can be written in terms of $\int_{\Delta_i} v \, \mathrm{d}x$ and $\int_{S_i} v \, \mathrm{d}s$. Use Cauchy's inequality and identities like

$$\int_{S_1} v \, \mathrm{d}s = \int_{\partial M} (v\, t \, , \, n)_{l_2} \, \mathrm{d}s = \int_M \mathrm{div}(v\, t) \, \mathrm{d}x \, , \quad \text{where} \quad t(x,y) = \begin{pmatrix} (x+1)/2 \\ 0 \end{pmatrix} \, .$$

30 Consider the assumptions and definitions in Example 28. Show, that there exists a constant $C > 0$ such that

$$\|v_h - v\|_{H^1(M)} \leq C\,|v|_{H^1(M)} \quad \forall v \in C^1(\overline{M})\,.$$

Hint: Show, that $v_h - v = \Pi_h v - v = \Pi_h(v + c) - (v + c)$ holds for any arbitrary constant function c. With Example 29 one obtains $\|v_h - v\|_{H^1(M)} \leq \tilde{C}\|v + c\|_{H^1(M)}$. In order to estimate $\|v + c\|_{H^1(M)}$ from above, use Poincare's inequality

$$\|w\|^2_{L_2(M)} \leq c_P^2 \left(\left(\int_M w \,\mathrm{d}x \right)^2 + |w|^2_{H^1(M)} \right)$$

for $w = v + c$, where c is chosen properly.

31* Consider the assumptions and definitions in Example 28 and replace $M = (-1,1) \times (-1,1)$ by $M_h = (-h,h) \times (-h,h)$, where $h \in (0,1]$. Show, that there exists a constant $c > 0$ independent of h ($c \neq c(h)$) such that

$$\|v_h\|_{H^1(M_h)} \leq c\,\|v\|_{H^1(M_h)} \quad \forall v \in C^1(\overline{M}_h)\ \forall h \in (0,1]\,.$$

Hint: Use $\|v_h\| \leq \|v_h - v\| + \|v\|$ and (after a proper transformation of variables) the estimate of Example 30.