T U T O R I A L

"Computational Mechanics"

to the lecture

"Numerical Methods in Continuum Mechanics 1"

Tutorial 05 Friday, April 30, 2010 (Time : $10^{\underline{15}}$ – $11^{\underline{00}}$, Room : HS 14)

4 Analysis and Numerics of Mixed Variational Problems

4.1 Mixed Variational Problems

Consider the mixed variational problem: Find $u \in X$ and $\lambda \in \Lambda$, such that

$$\begin{aligned} a(u,v) + b(v,\lambda) &= \langle f, v\rangle \quad \forall v \in X\,, \\ b(u,\mu) &= \langle g, \mu\rangle \quad \forall \mu \in \Lambda\,. \end{aligned}$$

In order to guarantee a unique existence of the solution (see Theorem 2.4 (*Brezzi*) in the lectures) one has to verify the following conditions:

1. The linear forms f and g are continuous, i.e.,

$$f \in X^*\,, \quad g \in \Lambda^*\,, \tag{4.15}$$

2. the bilinear forms $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ and $b(\cdot,\cdot) : X \times \Lambda \to \mathbb{R}$ are continuous, i.e., $\exists \alpha_2, \beta_2 = \text{const} > 0$:

$$|a(u,v)| \leq \alpha_2 \|u\|_X \|v\|_X \quad \forall u,\, v \in X\,, \tag{4.16}$$

$$|b(v,\mu)| \leq \beta_2 \|v\|_X \|\mu\|_\Lambda \quad \forall v \in X\,, \forall \mu \in \Lambda\,, \tag{4.17}$$

3. LBB (Ladyshenskaja – Babuska – Brezzi) condition: $\exists \beta_1 = \text{const} > 0$:

$$\inf_{\substack{\mu\in\Lambda \\ \mu\neq 0}} \sup_{\substack{v\in X \\ v\neq 0}} \frac{b(v,\mu)}{\|v\|_X \|\mu\|_\Lambda} \geq \beta_1\,, \tag{4.18}$$

4. Ker B–ellipticity, i. e., $\exists \alpha_1 = \text{const} > 0$:

$$a(v,v) \geq \alpha_1 \|v\|_X^2 \quad \forall v \in \operatorname{Ker} B\,, \tag{4.19}$$

where $\operatorname{Ker} B = \{v \in X \mid Bv = 0 \ \ (\text{in } \Lambda^*)\} = \{v \in X \mid \underbrace{b(v,\mu)}_{=\langle Bv\,,\,\mu\rangle} = 0 \ \ \forall \mu \in \Lambda\}$.

18 Consider the mixed formulation of the 1st BVP of the biharmonic equation (see Example 1.3 in the lectures, and Exercise 9 of the tutorials): Find $w \in X := H^1(\Omega)$ and $u \in \Lambda := H_0^1(\Omega)$ such that there holds

$$\begin{aligned} \int_\Omega w\, m \,\mathrm{d}x - \int_\Omega \nabla m\, \nabla u \,\mathrm{d}x &= 0 \quad \forall m \in X\,, \\ -\int_\Omega \nabla w\, \nabla v \,\mathrm{d}x &= \int_\Omega f\, v \,\mathrm{d}x \quad \forall v \in \Lambda\,, \end{aligned}$$

Show, that for this problem, the conditions (4.16) and (4.18) are satisfied ! What can you say about (4.19) ?

19 Consider the Stokes problem (see Example 1.1 in the lectures): Find $u \in X := [H_0^1(\Omega)]^3$ and $p \in \Lambda := \{q \in L_2(\Omega) \mid \int_\Omega q \,\mathrm{d}x = 0\}$ such that there holds

$$\begin{aligned} \frac{1}{\mathrm{Re}} \int_\Omega \nabla u : \nabla v \,\mathrm{d}x - \int_\Omega \operatorname{div} v\, p \,\mathrm{d}x &= \int_\Omega f\, v \,\mathrm{d}x \quad \forall v \in X\,, \\ -\int_\Omega \operatorname{div} u\, q \,\mathrm{d}x &= 0 \quad \forall q \in \Lambda\,, \end{aligned}$$

where the Reynolds number Re is positive, and where : denotes the inner product $A : B = \sum_{i,j=1}^3 a_{ij}\, b_{ij}$, defined for matrices $A = (a_{ij})_{i,j=1,2,3}$ and $B = (b_{ij})_{i,j=1,2,3}$. Show, that for this problem the conditions (4.16) – (4.19), except for the too difficult part (4.18), are satisfied.

20 Consider the mixed formulation of the Dirichlet problem for the Poisson equation (see Example 1.2 in the lectures): Find $\sigma \in X := H(\mathrm{div}, \Omega) = \{\tau \in [L_2(\Omega)]^3 \mid \operatorname{div} \tau \in L_2(\Omega)\}$ with the norm $\|\tau\|_X^2 = \|\tau\|_{L_2(\Omega)}^2 + \|\operatorname{div} \tau\|_{L_2(\Omega)}^2$ and $u \in \Lambda := L_2(\Omega)$ such that there holds

$$\begin{aligned} \int_\Omega \sigma^T \tau \,\mathrm{d}x + \int_\Omega \operatorname{div} \tau\, u \,\mathrm{d}x &= 0 \quad \forall \tau \in X\,, \\ \int_\Omega \operatorname{div} \sigma\, v \,\mathrm{d}x &= \int_\Omega f\, v \,\mathrm{d}x \quad \forall v \in \Lambda\,. \end{aligned}$$

Show, that for this problem the conditions (4.16) – (4.19) are satisfied.

Hint: In order to show (4.18), i. e.,

$$\exists \beta_1 > 0 : \inf_{\substack{v \in \Lambda \\ v \neq 0}} \sup_{\substack{\tau \in X \\ \tau \neq 0}} \frac{b(\tau, v)}{\|\tau\|_X \|v\|_\Lambda} \geq \beta_1\,, \quad \text{where } b(\tau, v) = \int_\Omega \operatorname{div} \tau\, v \,\mathrm{d}x\,,$$

you can use Example **21***: For an arbitrary $v \in \Lambda = L_2(\Omega)$ choose $\tau = -\nabla \mu$, where $\mu \in H_0^1(\Omega)$ solves the variational problem

$$\int_\Omega \nabla \mu^T \nabla \eta \,\mathrm{d}x = \int_\Omega v\, \eta \,\mathrm{d}x \quad \forall \eta \in H_0^1(\Omega)\,.$$

21* Let X and Λ be real Hilbert spaces and $B : X \to \Lambda^*$ a bounded linear operator. Show, that B satisfies the LBB-condition

$$\exists \beta_1 > 0 : \inf_{\substack{v \in \Lambda \\ v \neq 0}} \sup_{\substack{\tau \in X \\ \tau \neq 0}} \frac{\langle B\tau\,,\, v\rangle}{\|\tau\| \|v\|} \geq \beta_1\,,$$

if and only if there exists $c = \text{const} > 0$ such that for all $v^* \in \Lambda^*$ there exists a $\tau \in X$ such that $B\tau = v^*$ and $\|\tau\|_X \leq c\, \|v^*\|_{\Lambda^*}$.