

TUTORIAL

“Computational Mechanics”

to the lecture

“Numerical Methods in Continuum Mechanics 1”

Tutorial 03

Friday, March 26, 2010 (Time : 10¹⁵ – 11⁰⁰, Room : HS 14)

2 Theorem of Babuska and Aziz

Theorem 1.5: (Babuška und Aziz, 1972)

Let U and V be Hilbert spaces. Then the linear map (operator) $A : U \mapsto V^*$ is an isomorphism (bijective, A and A^{-1} continuous) if and only if (= iff) the corresponding bilinear form $a(.,.) : U \times V \rightarrow \mathbb{R}^1$ fulfills the following conditions:

1. continuity, i.e. $\exists \mu_2 = \text{const.} > 0$:

$$(8) \quad |a(u, v)| \leq \mu_2 \|u\|_U \|v\|_V \quad \forall u \in U, \forall v \in V,$$

2. inf-sup-condition, i.e. $\exists \mu_1 = \text{const.} > 0$:

$$(9) \quad \inf_{u \in U} \sup_{v \in V} \frac{a(u, v)}{\|u\|_U \|v\|_V} \geq \mu_1 = \text{const.} > 0,$$

3. $\forall v \in V \setminus \{0\} \exists u \in U :$

$$(10) \quad a(u, v) \neq 0.$$

Proof:

1. \Leftarrow **sufficiency**, i.e (8) - (9) $\implies A$ is an isomorphism !

- (8) $\iff A : U \mapsto V^*$ - continuous (= bounded), i.e. $A \in L(U, V^*)$ and $\|A\|_{L(U, V^*)} \leq \mu_2$.
- A is injective (= one-to-one mapping). Indeed, assuming $Au_1 = Au_2 \iff a(u_1, v) = a(u_2, v) \forall v \in V$ and using (9), we obtain

$$0 = \sup_{v \in V} \frac{a(u_1 - u_2, v)}{\|v\|_V} \geq \mu_1 \|u_1 - u_2\|_U,$$

i.e. $u_1 = u_2$.

- $A : U \mapsto A(U) \subset V^*$ yields $\exists A^{-1} : A(U) \mapsto U$.
- $A^{-1} : A(U) \mapsto U$ is continuous. Indeed,

$$\mu_1 \|u\|_U \leq \sup_{v \in V} \frac{\langle Au, v \rangle}{\|v\|_V} = \sup_{v \in V} \frac{\langle f, v \rangle}{\|v\|_V} = \|f\|_{V^*},$$

i.e. $\mu_1 \|A^{-1}f\|_U \leq \|f\|_{V^*} \quad \forall f \in A(U)$,

i.e. A^{-1} is bounded, i.e. A^{-1} is continuous.

- $A : U \leftrightarrow A(U)$, $A^{-1} : A(U) \leftrightarrow U$ - continuous $\implies A(U) = \overline{A(U)}$.
- Closed Range Theorem (Theorem 1.4) yields: $A(U) = \overline{A(U)} = (\ker A^*)^0 = \{0\}^0 = V^*$ due to (10). Indeed,
 $\ker A^* = \{v \in V : \langle A^*v, u \rangle_{U^* \times U} = \langle Au, v \rangle_{V^* \times V} = a(u, v) = 0 \forall u \in U\} = \{0\}$.

An alternative proof of the sufficiency follows from Exercises 13*. This results in a constructive proof !

2. \implies **necessity**: see Exercises 10 - 12 !

10 Show that the fact that $A \in L(U, V^*)$ is an isomorphism yields the continuity (= boundedness) of the bilinear form $a(u, v) = \langle Au, v \rangle$! Define the boundedness constant μ_2 of the bilinear form $a(., .)$!

11 Show that the fact that $A \in L(U, V^*)$ is an isomorphism yields the inf-sup-condition of the bilinear form $a(u, v) = \langle Au, v \rangle$! Define the inf-sup-constant μ_1 of the bilinear form $a(., .)$!

12 Show that the fact that $A \in L(U, V^*)$ is an isomorphism also yields the third condition of the Babuška-Aziz-Theorem 1.5 !

13* Let us assume that the sufficient conditions (8) - (9) of the Babuška-Aziz-Theorem 1.5 are fulfilled. Let us consider the variational problem: find $u \in U$ such that

$$A(u, v) = \langle F, v \rangle \quad \forall v \in V, \quad (2.10)$$

where the bilinear form $A(u, v)$ and the linear form $\langle F, v \rangle$ are defined by the identities

$$A(u, v) = \langle A^*JAu, v \rangle \quad \forall u, v \in U \quad (2.11)$$

and

$$\langle F, v \rangle = \langle A^*Jf, v \rangle \quad \forall v \in U, \quad (2.12)$$

respectively. Here $A^* : V \longrightarrow U^*$ denotes the adjoint to $A : U \longrightarrow V^*$ operator and $J : V^* \longrightarrow V$ is the Riesz isomorphism between the Hilbertspaces V^* and V . Show that the linear form $\langle F, . \rangle$ and bilinear form $A(., .)$ fulfil the assumption of the Lax-Milgram-Theorem and provide the ellipticity and the boundedness constants of the bilinear form $A(., .)$!