Let $\mathcal{T}_h = \{\delta_r : r \in \mathbb{R}_h\}$ be an admissible subdivision of a polygonal domain $\Omega \subset \mathbb{R}^2$ into acute triangles and let $\mathcal{T}_{\mathcal{H}} = \{\mathcal{H}(x) : x \in \overline{\omega}_h\}$ be the secondary mesh obtained by the PB method.
The Petrov-Galerkin method discussed in class for the boundary value problem

$$\begin{aligned} -\mathrm{div}\big(a(x)\,\mathrm{grad}\,u(x)\big) &= f(x) \qquad \forall x \in \Omega, \\ u(x) &= 0 \qquad \forall x \in \partial\Omega. \end{aligned}$$

leads to the following discrete variational problem.
Find $u_h \in V_{0h}$ such that

$$\bar{a}(u_h,\, \overline{v}) \;=\; (f,\, \overline{v})_{L^2(\Omega)} \qquad \forall \overline{v} \in T_{0h} \tag{11.1}$$

(with the notations introduced in class).

56 Show: the variational problem (11.1) can be written as the finite difference method

$$\begin{aligned} (L_h\, \underline{u}_h)(x) &= f_h(x) \qquad \forall x \in \omega_h\,, \\ \underline{u}_h(x) &= 0 \qquad \forall x \in \gamma_h \end{aligned}$$

for the grid function $\underline{u}_h : \overline{\omega}_h \to \mathbb{R}$, given by $\underline{u}_h(x) = u_h(x)$ for all $x \in \overline{\omega}_h$, with

$$(L_h\, v)(x) \;=\; -\frac{1}{H(x)} \sum_{\xi \in S'_h(x)} \bar{a}(x_\xi)\, \frac{v(\xi) - v(x)}{h(x,\, \xi)}\, s(x_\xi) \qquad \text{for all } x \in \omega_h\,,$$

$$\bar{a}(x_\xi) \;=\; \frac{1}{s(x_\xi)} \int_{\zeta(x,\,\xi)} a(y)\, ds_y\,, \qquad \text{and} \qquad f_h(x) \;=\; \frac{1}{H(x)} \int_{\mathcal{H}(x)} f(y)\, dy.$$

57 Assume the notations introduced in class and those introduced above.
Let the discrete Laplace operator Δ_h be given by

$$(\Delta_h\, v)(x) \;=\; \frac{1}{H(x)} \sum_{\xi \in S'_h(x)} \frac{v(\xi) - v(x)}{h(x,\, \xi)}\, s(x_\xi) \qquad \text{for all } x \in \omega_h\,,$$

and let us introduce the following discrete scalar products for the two grid functions $v : \overline{\omega}_h \to \mathbb{R}$ and $w : \overline{\omega}_h \to \mathbb{R}$ with $v(x) = w(x) = 0$ for all $x \in \gamma_h$:

$$\begin{aligned} (v,\, w)_{L^2(\omega_h)} &:= \sum_{x \in \omega_h} v(x)\, w(x)\, H(x), \\ (v,\, w)_{H^1_0(\omega_h)} &:= \sum_{x_\xi} \frac{[v(\xi) - v(x)][w(\xi) - w(x)]}{h(x,\, \xi)^2}\, H'(x_\xi). \end{aligned}$$

Show the identity

$$(-\Delta_h\, v,\, w)_{L^2(\omega_h)} \;=\; (v,\, w)_{H^1_0(\omega_h)}$$

58 Assume the notations introduced in class and in the previous exercises.
For $w_h \in V_{0h}$, let $\underline{w}_h$ denote the corresponding grid function $\underline{w}_h : \overline{\omega}_h \to \mathbb{R}$, given by $\underline{w}_h(x) = w_h(x)$ for all $x \in \overline{\omega}_h$.
Show the identity

$$(\underline{u}_h,\, \underline{v}_h)_{H^1_0(\omega_h)} \;=\; (\mathrm{grad}\, u_h,\, \mathrm{grad}\, v_h)_{L^2(\Omega)} \qquad \forall u_h,\, v_h \in V_{0h}\,.$$

Hints:

(i) Express $(\underline{u}_h,\, \underline{v}_h)_{H_0^1(\omega_h)}$ with the help of $\Delta_h\, \underline{u}_h$, see exercise $\boxed{57}$.

(ii) Express $\Delta_h\, \underline{u}_h$ with the help of the bilinear form $\bar{a}(\cdot,\, \cdot)$, see exercise $\boxed{56}$.

(iii) Express $\bar{a}(\cdot,\, \cdot)$ with the help of the bilinear form $a(\cdot,\, \cdot)$, given by

$$a(u,\, v) \;=\; (\operatorname{grad} u,\, \operatorname{grad} v)_{L^2(\Omega)}\,,$$

see class.

$\boxed{59}$ Assume the notations introduced in class and in the previous exercises.

For $w_h \in V_{0h}$, let $\underline{w}_h$ denote the corresponding grid function (see exercise $\boxed{58}$). In contrast to that, let $\overline{w}_h \in T_{0h}$ denote the corresponding piecewise constant function (with respect to the secondary mesh) with $\overline{w}_h(x) = w_h(x)$ for all $x \in \overline{\omega}_h$.

Show the identity

$$(\underline{u}_h,\, \underline{v}_h)_{L^2(\omega_h)} \;=\; (\overline{u}_h,\, \overline{u}_h)_{L^2(\Omega)} \qquad \forall u_h,\, v_h \in V_{0h}\,.$$

$\boxed{60}$ Assume the notations introduced in class and in the previous exercises.

Show that there exist constants $\underline{c},\, \overline{c} > 0$, independent of h, such that

$$\underline{c}\,(\overline{v}_h,\, \overline{v}_h)_{L^2(\Omega)} \;\le\; (v_h,\, v_h)_{L^2(\Omega)} \;\le\; (\overline{v}_h,\, \overline{v}_h)_{L^2(\Omega)} \qquad \forall v_h \in V_{0h}\,.$$

Hint: Transform to the reference triangle and observe that

$$\lambda_{\min}(\widehat{M})\,(\widehat{v},\, \widehat{v})_{\ell^2} \;\le\; (\widehat{M}\,\widehat{v},\, \widehat{v})_{\ell^2} \;\le\; \lambda_{\max}(\widehat{M})\,(\widehat{v},\, \widehat{v})_{\ell^2}\,,$$

where $\widehat{M}$ denotes the mass matrix on the reference triangle.

$\boxed{61}$ Assume the notations introduced in class and in the previous exercises.

Show the discrete Friedrichs inequality: there exists a constant $\widetilde{c}_F > 0$, independent of h, such that

$$\|\underline{v}\|_{L^2(\omega_h)} \;\le\; \widetilde{c}_F\, |\underline{v}|_{H_0^1(\omega_h)}$$

for all grid functions $\underline{v} : \overline{\omega}_h \to \mathbb{R}$ with $\underline{v}(x) = 0\ \forall x \in \gamma_h$. You can use the (continuous) Friedrichs inequality

$$\|v\|_{L^2(\Omega)} \;\le\; C_F\, |v|_{H^1(\Omega)} \qquad \forall v \in H_0^1(\Omega)$$

without proof.