50 Consider the boundary value problem

$$\begin{aligned} L\,u(x) &= f(x) \qquad \forall x \in (0,1), \\ u(0) = u(1) &= 0 \end{aligned}$$

with

$$L\,u(x) = -u''(x) + b\,u'(x)$$

and $b \in \mathbb{R}$. Let $\mathcal{T}_h$ denote the equidistant subdivision of the interval $[0,1]$ with mesh size $h = 1/n$, $n \in \mathbb{N}$ and let the corresponding set of nodes be $\overline{\omega}_h = \{x_0,\, x_1,\, \ldots,\, x_n\}$. Consider the following variational formulation of the boundary value problem: Find $u \in H_0^1(0,1)$ such that

$$a(u,v) = (f,v)_{L^2(0,1)} \qquad \forall v \in H_0^1(0,1) \tag{10.1}$$

with

$$a(u,v) = (u',v')_{L^2(0,1)} + (b\,u',v)_{L^2(0,1)}\,.$$

For $b = 0$, the finite element discretization by the Courant element is equivalent to the finite difference method

$$\begin{aligned} (L_h\,u_h)(x) &= f_h(x) \qquad \forall x \in \omega_h\,, \\ u_h(0) = u_h(1) &= 0 \end{aligned}$$

with

$$(L_h\,u_h)(x) = -\frac{1}{h^2}\big[u_h(x-h) - 2\,u_h(x) + u_h(x+h)\big]$$

and

$$f_h(x) = \frac{1}{h}\int_{x-h}^{x+h} f(y)\,p^{(x)}(y)\,dy,$$

where $p^{(x)}$ denotes the nodal basis function associated to the node $x \in \overline{\omega}_h$. This follows from the identity

$$\frac{1}{h}\,a(u_h,\, p^{(x)}) = (L_h\,u_h)(x),$$

which is easy to show.

Consider the case $b \neq 0$ and find the finite difference method that is equivalent to the finite element discretization by the Courant element.

51 Assume the notations of the previous example.

Let α_h be a given positive real number and consider the following discrete variational problem: find $u_h \in V_h$ such that

$$a_h(u_h,\, v_h) = (f,\, v_h)_{L^2(0,1)} \qquad \forall v_h \in V_h \tag{10.2}$$

with

$$a_h(u_h,\, v_h) = (u_h',\, v_h')_{L^2(0,1)} + (b\,u_h',\, v_h + \alpha_h\,b\,v_h')_{L^2(0,1)}\,,$$

where $V_h \subset H_0^1(0,1)$ denotes the finite element space given by the Courant element. Determine the parameter α_h such that the finite element discretization by the Courant element applied to (10.2) is equivalent to the following finite difference method:

$$\begin{aligned}(L_h\, u_h)(x) &= f_h(x) \qquad \forall x \in \omega_h\,,\\ u_h(0) = u_h(1) &= 0,\end{aligned}$$

with

$$\begin{aligned}(L_h\, u_h)(x) &= -\frac{1}{h^2}\big[u_h(x-h) - 2\,u_h(x) + u_h(x+h)\big] + \\ &+ b \left\{ \begin{array}{ll} \frac{1}{h}\big[u_h(x) - u_h(x-h)\big] & \text{if } b \geq 0 \\ \frac{1}{h}\big[u_h(x+h) - u_h(x)\big] & \text{if } b < 0 \end{array} \right\}.\end{aligned}$$

52 A finite difference method of the form

$$\begin{aligned}(L_h\, u_h)(x) &= f_h(x) \qquad \forall x \in \omega_h\,,\\ u_h(0) = u_h(1) &= 0\end{aligned}$$

with

$$(L_h\, u_h)(x_h) = A_h(x)\, u_h(x) - \sum_{\xi \in S'_h(x)} B_h(x,\xi)\, u_h(\xi)$$

is called *monotone* if and only if

(i) $A_h(x) > 0$ for all $x \in \omega_h$,

(ii) $B_h(x,\xi) > 0$ for all $\xi \in S'_h(x)$, $x \in \omega_h$,

(iii) $D_h(x) := A_h(x) - \sum_{\xi \in S'_h(x)} B_h(x,\xi) \geq 0$ for all $x \in \omega_h$.

Under which conditions on b and h are the finite difference methods for excercises 50 and 51 monotone?

53 Assume the notations of the previous exercises. Let $\{\delta_r\}_{r\in\mathbb{R}}$ denote the elements of the triangulation, i. e., the intervals $(x_i,\, x_{i+1})$.

Show that

$$a_h(u_h,\, v_h) = \sum_{r\in\mathbb{R}_h} a^{(r)}(u_h,\, v_h + \alpha_h\, b\, v'_h),$$

with the element-local bilinear forms

$$a^{(r)}(w,\, v) = (u',\, v')_{L^2(\delta_r)} + (b\, u',\, v)_{L^2(\delta_r)}\,.$$

54 Assume the notations of the previous exercises.

Show that

$$a_h(u_h,\, v_h) = a(u_h,\, v_h) + \sum_{r\in\mathbb{R}} \alpha_h\, (L\, u_h,\, L\, u_h)_{L^2(\delta_r)}\,.$$

55 Assume the notations of the previous exercises.

Consider the variational problem to find $u_h \in V_h$ such that

$$a_h(u_h,\, v_h) = \langle F_h,\, v_h\rangle \qquad \forall v_h \in V_h\,,$$

with

$$
\begin{aligned}
a_h(u_h,\, v_h) &= a(u_h,\, v_h) + \sum_{r\in\mathbb{R}_h} \alpha_h\,(Lu_h,\, L\,v_h)_{L^2(\delta_r)} \\
\langle F_h,\, v_h\rangle &= (f,\, v_h)_{L^2(0,1)} + \sum_{r\in\mathbb{R}_h} \alpha_h\,(f,\, L\,v_h)_{L^2(\delta_r)}.
\end{aligned}
$$

Let $u \in H_0^1(0,1)$ be the exact solution of (10.1).
Show that if $u \in H_0^1(0,1) \cap H^2(0,1)$, then

$$
\langle F_h,\, v_h\rangle - a_h(u, v_h) = 0 \qquad \forall v_h \in V_h\,.
$$