Numerical Methods for Elliptic Partial Differential Equations SS 2010
Tutorial 9 Tuesday, 8 June 2010 10.15–11.45, T 041

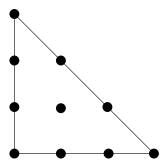
Read and understand:

Let $(\mathcal{T}_h)_{h\in\Theta}$ be a family of admissible subdivisions $\mathcal{T}_h = \{\delta_r : r \in \mathbb{R}_h\}$ of a bounded domain $\Omega \subset \mathbb{R}^2$ into triangles. The length of the longest edge of δ_r is denoted by $h^{(r)}$. Let Δ denote the reference triangle and x_{δ_r} the affine linear mapping from Δ to δ_r with its Jacobian J_{δ_r} (see Tutorial 7).
For each $m \in \mathbb{N}_0$, it can be shown that there exist constants c_1 and c_2 (depending only on m) such that

$$|v|_{H^m(\delta_r)} \leq c_1 \,|\det J_{\delta_r}|^{1/2}\, \|J_{\delta_r}^{-1}\|_{\ell^2}^m \,|v \circ x_{\delta_r}|_{H^m(\Delta)} \tag{9.1}$$

$$|v \circ x_{\delta_r}|_{H^m(\Delta)} \leq c_2 \,|\det J_{\delta_r}|^{-1/2}\, \|J_{\delta_r}\|_{\ell^2}^m \,|v|_{H^m(\delta_r)} \tag{9.2}$$

for all $h \in \Theta$, $r \in \mathbb{R}_h$ and $v \in H^m(\delta_r)$. In class (9.1) was shown for $m = 1$ and (9.2) for $m = 2$.
Consider the finite element space $V_h \subset H^1(\Omega)$, given by the shape functions $\mathcal{F}(\Delta) = P_k$, where $k \geq 1$, and the evaluations at all nodes $\xi^{(\alpha)} \in \{(\frac{i}{k}, \frac{j}{k}) : i, j \in \mathbb{N}_0 \text{ with } i + j \leq k\}$ as the nodal variables $\ell^{(\alpha)}$. For example, for $k = 3$:



In class, the linear (interpolation) operator $I_h : H^2(\Omega) \to V_h$ was constructed, where

$$(I_h(v))(x_{\delta_r}(\xi)) = (\hat{I}(v \circ x_{\delta_r}))(\xi) \qquad \forall \xi \in \Delta \quad \forall r \in \mathbb{R}_h\,,$$

with the corresponding linear (interpolation) operator $\hat{I} : H^2(\Delta) \to P_k$ on the reference element. For all integers s and l with $0 \leq s \leq l$ and $2 \leq l \leq k+1$ it can be shown that there exists a constant c_3 (depending only on s and l) with

$$|\widehat{v} - \hat{I}(\widehat{v})|_{H^s(\Delta)} \leq c_3\, |\widehat{v}|_{H^l(\Delta)} \qquad \forall \widehat{v} \in H^l(\Delta). \tag{9.3}$$

In class, (9.3) was shown for $s = 1$ and $l = 2$.
For $m \in \mathbb{N}_0$ consider the so-called *broken Sobolev space* $H^m(\Delta, \mathcal{T}_h)$, given by

$$H^m(\Delta, \mathcal{T}_h) = \{v \in L^2 : v_{|\delta_r} \in H^m(\delta_r) \quad \forall r \in \mathbb{R}_h\}$$

with semi-norm

$$|v|_{H^m(\Omega,\mathcal{T}_h)} = \Big(\sum_{r\in\mathbb{R}_h} |v|^2_{H^m(\delta_r)}\Big)^{1/2}.$$

Obviously,

$$H^m(\Omega) \subset H^m(\Omega, \mathcal{T}_h) \quad \text{and} \quad |v|_H^m(\Omega) = |v|_{H^m(\Omega,\mathcal{T}_h)} \quad \forall v \in H^m(\Omega).$$

Assume that there are constants c_4 and c_5 such that

$$\|J_{\delta_r}\|_{\ell^2} \leq c_4\, h^{(r)} \qquad \forall h \in \Theta,\ r \in \mathbb{R}_h\,, \tag{9.4}$$

$$\|J_{\delta_r}^{-1}\|_{\ell^2} \leq c_5\, \frac{1}{h^{(r)}} \qquad \forall h \in \Theta,\ r \in \mathbb{R}_h\,. \tag{9.5}$$

43 Use conditions (9.4)–(9.5) to show that, for all integers s and l with $0 \le s \le l$ and $2 \le l \le k+1$, there exists a constant c_6 (depending only on $c_1, \ldots, c_5$) such that

$$|v - I_h(v)|_{H^s(\Omega,\mathcal{T}_h)} \;\le\; c_6\, h^{l-s}\, |v|_{H^l(\Omega)} \qquad \forall v \in H^l(\Omega),$$

with $h = \max_{r \in \mathbb{R}_h} h^{(r)}$.

44 Show the following statements.

(a) For all integers s and l with $0 \le s \le 1$ and $2 \le l \le k+1$, there exists a constant c_7 such that

$$|v - I_h(v)|_{H^s(\Omega)} \;\le\; c_7\, h^{l-s}\, |v|_{H^l(\Omega)} \qquad \forall v \in H^l(\Omega).$$

(b) For all integers s and l with $0 \le s \le l$ and $2 \le l \le k+1$, there exists a constant c_8 (depending only on $c_1, \ldots, c_5$) such that

$$|I_h(v)|_{H^s(\Omega,\mathcal{T}_h)} \;\le\; c_8\, \|v\|_{H^l(\Omega)} \qquad \forall v \in H^l(\Omega).$$

Hint: $I_h(v) = I_h(v) - v + v$

(c) There exists a constant c_9 such that

$$\|v - I_h(v)\|_{L^2(\Omega)} + h\,|v - I_h(v)|_{H^1(\Omega)} \;\le\; c_9\, h^2\, |v|_{H^2(\Omega)} \qquad \forall v \in H^2(\Omega).$$

45 Show that there exists a constant c_{10} such that the inverse inequality

$$\|v_h\|_{H^1(\Omega)} \;\le\; c_{10}\, \frac{1}{h_{\min}}\, \|v_h\|_{L^2(\Omega)} \qquad \forall v \in V_h$$

holds, where $h_{\min} := \min_{r \in \mathbb{R}_h} h^{(r)}$.

46* (BONUS example)
Does a positive constant c_{11} exist such that the "inverse" of the inverse inequality

$$\frac{1}{h_*}\, \|v_h\|_{L^2(\Omega)} \;\le\; c_{11}\, \|v_h\|_{H^1(\Omega)} \qquad \forall v \in V_h$$

(with $h_* = h$ or $h_* = h_{\min}$) is satisfied?

Programming

Continue the program from Tutorial 8. As a concrete example we consider the problem to find $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u(x) \cdot \nabla v(x) + u(x)\, v(x)\, dx \;=\; \int_\Omega f(x)\, v(x)\, dx \qquad \forall v \in H^1(\Omega), \tag{9.6}$$

with $f(x_1, x_2) = (5\pi^2 + \frac{1}{4}) \cos(2\pi\, x_1) \cos(4\pi\, x_2)$.

(please turn over)

47 Download `cg.hh`. Implement a Jacobi preconditioner:

```
class JacobiPreconditioner
{
public:
  JacobiPreconditioner (const SparseMatrix& K);
  void solve (const Vector& r, Vector& z);
};
```

Assemble the finite element system $K\,u = b$ for (9.6) for the initial mesh from `meshdemo.cc` and solve it using conjugate gradients with your Jacobi preconditioner. Solve the same system for the uniformly refined meshes with $h/h_0 = 2, 4, 8, 16$ where h_0 is the mesh size of the initial mesh.

You can visualize solutions calling `mesh.matlabOutput ("output.m", u);` from your program, and then loading the file into `matlab` (provided you have the PDE Toolbox).

48 Write a function

```
double calcElErrorL2 (const Point2D& p0, const Point2D& p1,
                      const Point2D& p2, ScalarField exact,
                      double v0, double v1, double v2);
```

that approximates the element L^2-error $\|v - v_h\|_{L^2(\delta_r)}$, where `exact`$=v$ and $v_h(x_{\delta_r}(\xi)) = \sum_{\alpha \in A} v^{(r,\alpha)}\, p^{(\alpha)}(\xi)$ with `v0`$=v^{(r,1)}$ etc.

Hint: Use the quadrature rule from Exercise 29 to approximate

$$\|v - v_h\|^2_{L^2(\delta_r)} \;=\; \int_{\delta_r} |v(x) - v_h(x)|^2\, dx \;=\; \int_{\Delta} |v(x_{\delta_r}(\xi)) - v_h(x_{\delta_r}(\xi)|^2\, |\det J_{\delta_r}|\, d\xi$$

49 Write a function

```
double calcErrorL2 (const Mesh& mesh, ScalarField exact,
                    const Vector& solution);
```

that approximates the global L^2-error $\|v - v_h\|_{L^2(\Omega)}$, where `exact`$=v$ and `solution`$=v_h$.

Hint: use `calcElErrorL2` in a loop over all elements.

Show that $u(x_1, x_2) = \frac{1}{4}\cos(2\pi\, x_1)\,\cos(4\pi\, x_2)$ is the unique solution of (9.6). Compute $\|u - u_h\|_{L^2(\Omega)}$ for each the finite element solution u_h from Exercise 47 for the different meshes.