Let $(\mathcal{T}_h)_{h\in\Theta}$ be a family of subdivisions $\mathcal{T}_h = \{\delta_r : r \in \mathbb{R}_h\}$ of a domain $\Omega \subset \mathbb{R}^2$ into non-degenerate triangles. For a triangle $\delta_r \in \mathcal{T}_h$ we denote by

- $x^{(r,\alpha)}$ its vertices,
- $\theta^{(r,\alpha)}$ the corresponding angles,
- $h^{(r,\alpha)}$ the length of the edge opposite to vertex $x^{(r,\alpha)}$,

where $\alpha \in A = \{1,2,3\}$. We define

$$h^{(r)} := \operatorname{diam}(\delta_r) = \max\{h^{(r,\alpha)} : \alpha \in A\},$$

and we denote the diameter of the largest inscribed circle by $\rho^{(r)}$. Furthermore, let

$$\Delta = \{\xi \in \mathbb{R}^2 : \xi_1 > 0,\ \xi_2 > 0,\ \xi_1 + \xi_2 < 1\}$$

be the reference triangle, and $x_{\delta_r} : \mathbb{R}^2 \to \mathbb{R}^2$ the map from Δ to δ_r, given by

$$x_{\delta_r}(\xi) \;=\; x^{(r,1)} + J_{\delta_r}\,\xi, \qquad \text{with} \quad J_{\delta_r} \;=\; \left(x^{(r,2)} - x^{(r,1)} \;\middle|\; x^{(r,2)} - x^{(r,1)}\right) \quad \in \mathbb{R}^{2\times 2}.$$

31 Show that the following statements are equivalent:

(i) there exists a constant $\theta_0 > 0$ such that
$\theta_0 \;\le\; \theta^{(r,\alpha)}$ for all $h \in \Theta$, $r \in \mathbb{R}_h$, $\alpha \in A$

(ii) there exists a constant $\sigma_0 > 0$ such that
$\sigma_0\, h^{(r)} \;\le\; \rho^{(r)}$ for all $h \in \Theta$, $r \in \mathbb{R}_h$.

Hint: Show and use that

$$\frac{\rho^{(r)}}{2}\left[\cot\left(\tfrac{1}{2}\,\theta^{(r,\beta)}\right) + \cot\left(\tfrac{1}{2}\,\theta^{(r,\gamma)}\right)\right] \;=\; h^{(r,\alpha)}$$

for pairwise distinct indices α, β, $\gamma \in A$.

32 Show that there exist constants $\underline{c}_1$, $\overline{c}_1 > 0$ such that

$$\underline{c}_1\,\rho^{(r)}\,h^{(r)} \;\le\; |\det J_{\delta_r}| \;\le\; \overline{c}_1\,\rho^{(r)}\,h^{(r)} \qquad \forall h \in \Theta,\ r \in \mathbb{R}_h\,.$$

Hint: Show and use that $2\,\mathrm{meas}(\delta_r) = \left(h^{(r,1)} + h^{(r,2)} + h^{(r,3)}\right)\rho^{(r)}$.

33 Show that there exist constants c_2, $c_3 > 0$ such that

$$\|J_{\delta_r}\|_{\ell^2} \;\le\; c_2\,h^{(r)} \qquad \text{and} \qquad \|J_{\delta_r}^{-1}\|_{\ell^2} \;\le\; c_3\,\frac{1}{\rho^{(r)}} \qquad \forall h \in \Theta,\ r \in \mathbb{R}_h\,,$$

with the spectral matrix $\|M\|_{\ell^2} := \sup_{v\neq 0} \frac{\|Mv\|_{\ell^2}}{\|v\|_{\ell^2}}$ and the Euclidean norm $\|v\|_{\ell^2}$.

Hint: Show and use that

$$\|J_{\delta_r}\|_{\ell^2} \;=\; \sup_{\xi\in\partial B_R(\eta)} \frac{\|x_{\delta_r}(\xi) - x_{\delta_r}(\eta)\|_{\ell^2}}{\|\xi - \eta\|_{\ell^2}} \qquad \forall \eta \in \mathbb{R}^2,\ \forall R > 0,$$

with $\partial B_R(\eta) = \{\xi \in \mathbb{R}^2 : \|\xi - \eta\|_{\ell^2} = R\}$. In particular, consider the larges inscribed circle of Δ for $\partial B_R(\eta)$ in order to show the first upper bound.

34 Show that a family of triangulations (subdivisions into non-degenerate triangles) is regular if and only if it is quasi-uniform.

35 Show that a family $(\mathcal{T}_h)_{h\in\Theta}$ of subdivisions is shape-regular if and only if there exist positive constants $\underline{c}_1'$, $\overline{c}_1'$, c_2', and c_3' such that

(i) $\underline{c}_1'\,(h^{(r)})^2 \;\le\; |\det J_{\delta_r}| \;\le\; \overline{c}_1'\,(h^{(r)})^2$ for all $h \in \Theta$, $r \in \mathbb{R}_h$,

(ii) $\|J_{\delta_r}\|_{\ell^2} \;\le\; c_2'\,h^{(r)}$ for all $h \in \Theta$, $r \in \mathbb{R}_h$, and

(iii) $\|J_{\delta_r}^{-1}\|_{\ell^2} \;\le\; c_3'\,(h^{(r)})^{-1}$ for all $h \in \Theta$, $r \in \mathbb{R}_h$.

36 Assume that the family $(\mathcal{T}_h)_{h\in\Theta}$ is shape-regular and denote by V_h the corresponding finite element space of the Courant element.

Show that there exist positive constants $\underline{\boldsymbol{c}}_1$ and $\overline{\boldsymbol{c}}_1$ such that

$$\underline{\boldsymbol{c}}_1 \left(\min_{r\in\mathbb{R}_h} h^{(r)}\right)^d (\underline{v}_h,\, \underline{v}_h)_{\ell^2} \;\le\; (v_h,\, v_h)_{H^1(\Omega)} \;\le\; \overline{\boldsymbol{c}}_1 \left(\max_{r\in\mathbb{R}_h} h^{(r)}\right)^{d-2} (\underline{v}_h,\, \underline{v}_h)_{\ell^2} \qquad \forall v_h \in V_h\,,$$

with $d = 2$.
Hint: Modify the proof from the lecture accordingly.