**19** (a) Consider the isoparametric bilinear element on a non-degenerate quadrilateral, given by the reference domain $\Delta = [-1,1]^2$, the space of shape functions

$$\mathcal{F}(\Delta) \;=\; \mathrm{span}\{1, \xi_1, \xi_2, \xi_1\,\xi_2\} \;=\; Q_1,$$

and the nodal variables defined by the function evaluations at the four vertices of $\Delta$. Let $\delta$ denote the element domain obtained by rotating $\Delta$ by $45°$ counterlockwise around the origin. The corresponding (affine linear) transformation is denoted by $x_\delta$. Show that

$$\mathcal{F}(\delta) := \{v = \widehat{v} \circ x_\delta^{-1} : \widehat{v} \in \mathcal{F}(\Delta)\} \;=\; \mathrm{span}\{1, x_1, x_2, x_1^2 - x_2^2\},$$

and thus, $\mathcal{F}(\delta) \neq Q_1$.

(b) As an alternative, consider $\mathcal{F}(\delta) = Q_1$. Is such a shape function uniquely determined by its values at the four vertices of $\delta$?

**20** Let $\delta \subset \mathbb{R}^2$ be a general non-degenerate quadrilateral obtained from the reference element $\Delta = [-1,1]^2$ by a bilinear transformation $x_\delta$ which maps the vertices of $\Delta$ to the vertices of $\delta$.

The so-called rotated bilinear finite element on $\delta$ is given by the space of shape functions

$$\mathcal{F}(\delta) := \{v = \widehat{v} \circ x_\delta^{-1} : \widehat{v} \in \mathcal{F}(\Delta)\}$$

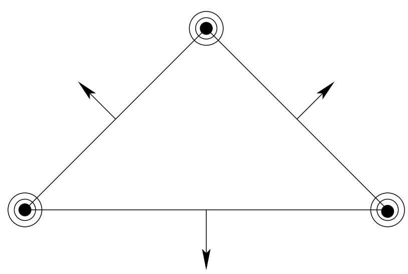
with

$$\mathcal{F}(\Delta) := \mathrm{span}\{1, \xi_1, \xi_2, \xi_1^2 - \xi_2^2\}.$$

and the nodal variables defined by the function evaluations at the midpoints of the four edges of $\delta$. Show that these nodal variables uniquely determine a shape function.

**21** Show that the rotated bilinear finite element, as defined in exercise **20** is not a $C^0$-element on admissible quadrilateral subdivisions.

**22** Show that the Argyris element



is a $C^1$-element on admissible triangulations.

*Hint:* Show that the values of a shape function and its normal derivative along an edge are uniquely determined by the prescribed values on this edge.

23 Consider the Raviart-Thomas element, given by

(i) a non-degenerate tetrahedron $\delta \subset \mathbb{R}^3$, whose midpoints and outwards unit normal vectors of the four faces are denoted by $x^{(\alpha)}$ and $n^{(\alpha)}$, respectively,

(ii) the (vector-valued) space of shape functions

$$\mathcal{F}(\delta) \;=\; \{v(x) = a + b\,x : a \in \mathbb{R}^3, b \in \mathbb{R}\},$$

and

(iii) the nodal variables

$$\ell^{(\alpha)}(v) \;=\; v(x^{(\alpha)}) \cdot n^{(\alpha)}.$$

Show property (b) of exercise 13.

*Hint:* First show that, if $\ell^{(\alpha)}(v) = 0$, then $v(x) \cdot n^{(\alpha)} = 0$ on the entire face containing $x^{(\alpha)}$. Observe that each vertex of the tetrahedron belongs to three different faces.

24 Consider the Nédélec element, given by

(i) a non-degenerate tetrahedron $\delta \subset \mathbb{R}^3$, whose midpoints of and unit vector along the six edges are denoted by $x^{(\alpha)}$ and $t^{(\alpha)}$, respectively,

(ii) the (vector-valued) space of shape functions

$$\mathcal{F}(\delta) \;=\; \{v(x) = a + b \times x : a, b \in \mathbb{R}^3\},$$

and

(iii) the nodal variables

$$\ell^{(\alpha)}(v) \;=\; v(x^{(\alpha)}) \cdot t^{(\alpha)}.$$

Show property (b) of exercise 13.

*Hint:* First show that, if $\ell^{(\alpha)}(v) = 0$, then $v(x) \cdot t^{(\alpha)} = 0$ on the entire edge containing $x^{(\alpha)}$. Observe that each vertex of the tetrahedron belongs to three different edges.