13 Let

(i) $\delta \subset \mathbb{R}^d$ be a domain with piecewise smooth boundary (the *element domain*),
(ii) $\mathcal{F}(\delta)$ be a finite-dimensional space of functions on δ (the *shape functions*), and
(iii) $\{\ell^{(\alpha)} : \alpha \in A\}$ be a finite set of linear functionals on $\mathcal{F}(\delta)$ (the *nodal variables*).

Let $\mathcal{F}(\delta)$ be an $|A|$-dimensional vector space, where $|A|$ is the cardinality of the index set A. Show that the following three statements are equivalent:

(a) $\{\ell^{(\alpha)} : \alpha \in A\}$ is a basis of $\mathcal{F}(\delta)^*$ (the dual space of $\mathcal{F}(\delta)$).
(b) For all $v \in \mathcal{F}(\delta)$: if $\ell^{(\alpha)}(v) = 0$ for all $\alpha \in A$, then $v(x) = 0$ for all $x \in \delta$.
(c) For any choice of values $v^{(\alpha)} \in \mathbb{R}$, $\alpha \in A$, there exists a unique function $v \in \mathcal{F}(\delta)$ with $\ell^{(\alpha)}(v) = v^{(\alpha)}$ for all $\alpha \in A$.

Hint: Fix a basis $\{p^{(\alpha)} : \alpha \in A\}$ of $\mathcal{F}(\delta)$ and show that each of the three statements corresponds to a property of the square matrix $M = \left(\ell^{(\alpha)}(p^{(\beta)})\right)_{\alpha,\beta \in A}$.

14 For $a \in \mathbb{R}^d \setminus \{0\}$ and $b \in \mathbb{R}$, let $L : \mathbb{R}^d \to \mathbb{R}$ be given by

$$L(x) \;=\; a \cdot x + b.$$

The set $\ker(L) := \{x \in \mathbb{R}^d : L(x) = 0\}$ is called a *hyperplane.* Let $p \in P_k$ (the set of polynomials in x of degree $\le k$).

Show that if p vanishes on the hyperplane $\ker(L)$, then we can write $p(x) = L(x)\,q(x)$, for some polynomial $q \in P_{k-1}$.

Hint: First, show the statement for the special hyperplane $x_d = 0$. For the general case, use that there exists an affine linear coordinate transformation $\widehat{x} = T(x)$ with $\widehat{x}_d = L(x)$, i.e., T transforms $\ker(L)$ to the hyperplane given by $\widehat{x}_d = 0$.

15 Consider the Courant finite element, given by

(i) the *element domain* δ being a non-degenerate triangle with vertices $x^{(\alpha)}$ for $\alpha \in A = \{1, 2, 3\}$,
(ii) the space of *shape functions* $\mathcal{F}(\delta) = P_1$, and
(iii) the nodal variables $\ell^{(\alpha)}$, given by

$$\ell^{(\alpha)}(v) \;=\; v(x^{(\alpha)}) \qquad \text{for } \alpha \in A \text{ and } v \in \mathcal{F}(\delta).$$

Use exercise **14** to show property (b) of exercise **13**.

Hint: Use the hyperplane (straight line) determined by an edge of the triangle to show that $v = L\,q$ with $q \in P_0$.

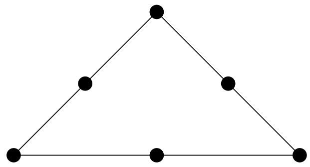
16 Consider the quadratic triangular Lagrange finite element, given by

(i) the *element domain* δ being a non-degenerate triangle, where we denote its vertices by $x^{(\alpha)}$ for $\alpha \in \{1, 2, 3\}$, and the midpoints of its edges by $x^{(\alpha)}$ for $\alpha \in \{4, 5, 6\}$,

(ii) the space of *shape functions* $\mathcal{F}(\delta) = P_2$, and

(iii) the nodal variables $\ell^{(\alpha)}$ for $\alpha \in A = \{1, 2, 3, 4, 5, 6\}$ given by

$$\ell^{(\alpha)}(v) = v(x^{(\alpha)}) \qquad \text{for } \alpha \in A \text{ and } v \in \mathcal{F}(\delta).$$



Use exercise 14 to show property (b) of excercise 13.

Hint: Use the same technique as in the previous exercise consecutively for two edges.

17 Consider the following quadratic triangular finite element, given by

(i) the *element domain* δ being a non-degenerate triangle, where we denote its vertices by $x^{(\alpha)}$ for $\alpha \in \{1, 2, 3\}$, and its edges by $e^{(\beta)}$ for $\beta \in \{1, 2, 3\}$,

(ii) the space of *shape functions* $\mathcal{F}(\delta) = P_2$, and

(iii) the nodal variables $\ell^{(\alpha)}$ given by

$$\ell^{(\alpha)}(v) = \left\{ \begin{array}{cc} v(x^{(\alpha)}) & \text{for } \alpha \in \{1,2,3\} \\ \displaystyle\int_{e^{(\alpha-3)}} v(x)\, ds_x & \text{for } \alpha \in \{4,5,6\} \end{array} \right\} \quad \text{for } v \in \mathcal{F}(\delta).$$

Use exercise 14 to show property (b) of excercise 13.

Hint: Use the same technique as in the previous exercise consecutively for two edges. Note that if for an open set e with $\ker(L) \cap e = \emptyset$, then L never changes sign on e.

18 Consider the cubic Hermite finite element, given by

(i) a non-degenerate triangle δ as *element domain*, whose vertices are denoted by $x^{(\alpha)}$, for $\alpha \in \{1, 2, 3\}$ and whose centroid is denoted by $x^{(4)}$,

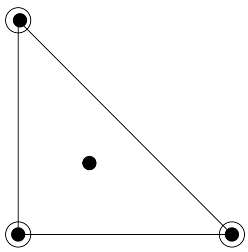
(ii) the space of *shape functions* $\mathcal{F}(\delta) = P_3$, and

(iii) the nodal variables $\ell^{(\alpha)}$ for $\alpha \in A = \{1, 2, \ldots, 10\}$ given by

$$\ell^{(\alpha)}(v) = v(x^{(\alpha)}) \qquad \text{for } \alpha \in \{1,2,3,4\}$$

and

$$\ell^{(3+2\beta)}(v) = \frac{\partial v}{\partial x_1}(x^{(\beta)}), \quad \ell^{(4+2\beta)}(v) = \frac{\partial v}{\partial x_2}(x^{(\beta)}) \quad \text{for } \beta \in \{1,2,3\}.$$



For $\delta = \Delta = \{\xi \in \mathbb{R}^2 : \xi_1 > 0,\ \xi_2 > 0,\ \xi_1 + \xi_2 < 1\}$, find an *explicit* formula for the *nodal basis function* $p^{(1)}(\xi)$ associated with the node $\xi^{(1)} = (0, 0)$.