Numerical Methods for Elliptic Partial Differential Equations SS 2010
Tutorial 3 Tuesday, 23 March 2010 10.15–11.45, T 041

The following exercises are around the space $H(\text{curl}, \Omega)$. Let us fix the domain $\Omega \subset \mathbb{R}^3$ with sufficiently smooth boundary $\partial\Omega$ and recall that

$$H(\text{curl}, \Omega) := \{\boldsymbol{v} \in L^2(\Omega)^3 : \text{curl}\, \boldsymbol{v} \in L^2(\Omega)^3\}$$

with $\boldsymbol{w} = \text{curl}\, \boldsymbol{v}$ (the weak curl) defined by the relation

$$\int_\Omega \boldsymbol{w} \cdot \boldsymbol{\varphi}\, dx \;=\; \int_\Omega \boldsymbol{v} \cdot \text{curl}\, \boldsymbol{\varphi}\, dx \qquad \forall \boldsymbol{\varphi} \in C_0^\infty(\Omega)^3.$$

We use the norm

$$\|\boldsymbol{u}\|_{H(\text{curl},\,\Omega)} := \left(\|\boldsymbol{u}\|^2_{L^2(\Omega)} + \|\text{curl}\, \boldsymbol{u}\|^2_{L^2(\Omega)}\right)^{1/2}.$$

Furthermore, recall that

$$H_0(\text{curl}, \Omega) := \big\{\boldsymbol{v} \in H(\text{curl}, \Omega) : \boldsymbol{v} \times \boldsymbol{n} = 0 \text{ on } \partial\Omega\big\} = \overline{C_0^\infty(\Omega)^3}^{\|\cdot\|_{H(\text{curl},\,\Omega)}}.$$

We also define the weak divergence $w = \text{div}\, \boldsymbol{v}$ by the relation

$$\int_\Omega w\, \varphi\, dx \;=\; \int_\Omega \boldsymbol{v} \cdot \text{grad}\, \varphi\, dx \qquad \forall \varphi \in C_0^\infty(\Omega),$$

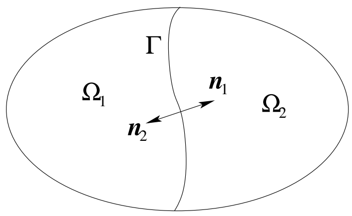
and the space $H(\text{div}, \Omega) := \{\boldsymbol{v} \in L^2(\Omega)^3 : \text{div}\, \boldsymbol{v} \in L^2(\Omega)\}$.

07 Assume that there are two disjoint open subdomains Ω_1 and Ω_2 such that

$$\overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\Omega}.$$

Let $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ the unit normal vectors outwards to Ω_1 and Ω_2, respectively, and define the interface $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$. We fix the two functions $\boldsymbol{v}_1 \in C^1(\Omega_1)^3$ and $\boldsymbol{v}_2 \in C^1(\Omega_2)^3$. Obviously, $\boldsymbol{v}_1 \in H(\text{curl}, \Omega_1)$ and $\boldsymbol{v}_2 \in H(\text{curl}, \Omega_2)$. Show that the function $\boldsymbol{v}^* \in L^2(\Omega)^3$ defined by

$$\boldsymbol{v}^*(x) = \begin{cases} \boldsymbol{v}_1(x) & \text{if } x \in \Omega_1 \\ \boldsymbol{v}_2(x) & \text{if } x \in \Omega_2, \end{cases}$$



satisfies $\boldsymbol{v}^* \in H(\text{curl}, \Omega)$ if and only if

$$\int_\Gamma \boldsymbol{v}_1 \times \boldsymbol{n}_1 + \boldsymbol{v}_2 \times \boldsymbol{n}_2\, ds = 0$$

(the tangential component of $\boldsymbol{v}^*$ is continuous).

08 Recall from exercise **02** that for sufficiently smooth functions $\boldsymbol{u}$ and $\boldsymbol{v}$,

$$\int_\Omega \text{curl}\, \boldsymbol{u} \cdot \boldsymbol{v}\, dx \;=\; \int_\Omega \boldsymbol{u} \cdot \text{curl}\, \boldsymbol{v}\, dx - \int_{\partial\Omega} (\boldsymbol{u} \times \boldsymbol{n}) \cdot \boldsymbol{v}\, ds$$

Show that if we define for $\boldsymbol{u} \in H(\text{curl}, \Omega)$ the operator γ_t by

$$\langle \gamma_t \boldsymbol{u}, \boldsymbol{v} \rangle \;=\; \int_\Omega \boldsymbol{u} \cdot \text{curl}\, \boldsymbol{v}\, dx - \int_\Omega \text{curl}\, \boldsymbol{u} \cdot \boldsymbol{v}\, dx \qquad \forall \boldsymbol{v} \in H^1(\Omega)^3,$$

then there exists a constant C such that

$$\langle \gamma_t \boldsymbol{u}, \boldsymbol{v} \rangle \;\le\; C\, \|\boldsymbol{u}\|_{H(\text{curl},\,\Omega)}\, \|\boldsymbol{v}\|_{H^1(\Omega)}.$$

Note: Using this estimate, one can show that the tangential trace $\boldsymbol{u} \times \boldsymbol{n}$ is well-defined for $\boldsymbol{u} \in H(\text{curl}, \Omega)$ as a functional in the dual of $H^{1/2}(\partial\Omega)^3$, where $H^{1/2}(\partial\Omega)$ is the trace space of $H^1(\Omega)$.

Consider the variational formulation to find $\boldsymbol{u} \in H_0(\mathrm{curl}, \Omega)$:

$$\underbrace{\int_\Omega \frac{1}{\mu} \,\mathrm{curl}\, \boldsymbol{u} \cdot \mathrm{curl}\, \boldsymbol{v} + \sigma\, \boldsymbol{u} \cdot \boldsymbol{v} \, dx}_{=a(\boldsymbol{u},\boldsymbol{v})} \;=\; \underbrace{\int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} \, dx}_{=\langle F, \boldsymbol{v}\rangle} \qquad \forall \boldsymbol{v} \in H_0(\mathrm{curl}, \Omega), \tag{3.1}$$

with constant parameters $\mu > 0$ and $\sigma \geq 0$.

09 Show that if $\sigma > 0$, then the bilinear form is $H_0(\mathrm{curl}, \Omega)$-coercive and -bounded, and that the linear functional F is $H_0(\mathrm{curl}, \Omega)$-bounded.

10 Consider the case $\sigma = 0$.

(a) Show that for all $p \in H_0^1(\Omega)$:

$$\mathrm{curl}\, \mathrm{grad}\, p = 0 \qquad \text{weakly in } \Omega.$$

Hint: Use the definitions of the weak curl and gradient and that $\mathrm{div}\, \mathrm{curl}\, \boldsymbol{\varphi} = 0$ for smooth functions $\boldsymbol{\varphi}$.

(b) Show that $\mathrm{grad}\, p \times \boldsymbol{n} = 0$ for all $p \in H_0^1(\Omega)$.
Hint: Start with $0 = \int_\Omega p \, \mathrm{div}\, \mathrm{curl}\, \boldsymbol{\varphi} \, dx$ and use integration by parts twice.
Note: (a) and (b) together imply that $\mathrm{grad}\, p \in H_0(\mathrm{curl}, \Omega)$ for all $p \in H_0^1(\Omega)$.

(c) Show (assuming that $\sigma = 0$) that

$$a(\mathrm{grad}\, p, \boldsymbol{v}) = 0 \qquad \forall p \in H_0^1(\Omega),\ \forall \boldsymbol{v} \in H_0(\mathrm{curl}, \Omega).$$

11 Consider again the case $\sigma = 0$. Show that if a solution to problem (3.1) exists, then

$$\int_\Omega \boldsymbol{J} \cdot \mathrm{grad}\, p \, dx = 0 \qquad \forall p \in H_0^1(\Omega).$$

Show that this implies

$$\mathrm{div}\, \boldsymbol{J} = 0 \quad \text{weakly in } \Omega.$$

12 Consider again the case $\sigma = 0$. Show that the bilinear form $a(\cdot, \cdot)$ is V_0-coercive where

$$V_0 := \left\{ \boldsymbol{v} \in H_0(\mathrm{curl}, \Omega) : \int_\Omega \boldsymbol{v} \cdot \mathrm{grad}\, p \, dx = 0 \quad \forall p \in H_0^1(\Omega) \right\}.$$

Use the "Friedrichs type inequality"

$$\|\boldsymbol{v}\|_{L^2(\Omega)} \;\leq\; C_F \, \|\mathrm{curl}\, \boldsymbol{v}\|_{L^2(\Omega)} \qquad \forall \boldsymbol{v} \in V_0,$$

which holds provided that the domain is simply connected and has a connected boundary (cf. Monk, 2003, Corollary 3.51).