01 Consider the linear differential operator L,

$$(Lu)(x) \;=\; \sum_{i,j=1}^{d} a_{ij}(x)\,\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{j=1}^{d} b_j(x)\,\frac{\partial u}{\partial x_j}(x) + c(x)\,u(x),$$

where $x \in \Omega \subset \mathbb{R}^d$.

(a) Show (under sufficient smoothness assumptions on u) that we can assume without loss of generality that $a_{ij} = a_{ji}$.

(b) Show that if $a_{ij} \in C^1(\Omega)$ then there exist coefficients $\widetilde{a}_{ij}$, $\widetilde{b}_j$ and $\widetilde{c}$ (specify them!) such that

$$(Lu)(x) \;=\; \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\Big(\sum_{j=1}^{d} \widetilde{a}_{ij}(x)\frac{\partial u}{\partial x_j}(x)\Big) + \sum_{j=1}^{d} \widetilde{b}_j(x)\,\frac{\partial u}{\partial x_j} + \widetilde{c}(x)\,u(x).$$

02 Let Ω be a bounded domains with sufficiently smooth boundary $\partial\Omega$ and outwards unit normal vector $\boldsymbol{n}$ on $\partial\Omega$. Furthermore, let $\varphi : \overline{\Omega} \to \mathbb{R}$, $\psi : \overline{\Omega} \to \mathbb{R}$, $\boldsymbol{u} : \overline{\Omega} \to \mathbb{R}^3$, and $\boldsymbol{v} : \overline{\Omega} \to \mathbb{R}^3$ be sufficiently smooth functions. Complete the following identities by integration by parts:

$$\int_\Omega \big(\operatorname{grad}\varphi(x)\big)\,\psi(x)\,dx \;=\; -\int_\Omega \varphi(x)\,\ldots\,dx + \int_{\partial\Omega} \varphi(x)\,\ldots\,ds$$
$$\int_\Omega \big(\operatorname{grad}\varphi(x)\big)\cdot\boldsymbol{v}(x)\,dx \;=\; -\int_\Omega \varphi(x)\,\ldots\,dx + \int_{\partial\Omega} \varphi(x)\,\ldots\,ds$$
$$\int_\Omega \big(\operatorname{div}\boldsymbol{u}(x)\big)\,\psi(x)\,dx \;=\; -\int_\Omega \boldsymbol{u}(x)\,\ldots\,dx + \int_{\partial\Omega} \boldsymbol{u}(x)\,\ldots\,ds$$
$$\int_\Omega \big(\operatorname{div}\boldsymbol{u}(x)\big)\cdot\boldsymbol{v}(x)\,dx \;=\; -\int_\Omega \ldots\,\boldsymbol{u}(x)\,\ldots\,dx + \int_{\partial\Omega} \boldsymbol{u}(x)\,\ldots\,ds$$
$$\int_\Omega \big(\operatorname{curl}\boldsymbol{u}(x)\big)\,\psi(x)\,dx \;=\; -\int_\Omega \boldsymbol{u}(x)\,\ldots\,dx + \int_{\partial\Omega} \boldsymbol{u}(x)\,\ldots\,ds$$
$$\int_\Omega \big(\operatorname{curl}\boldsymbol{u}(x)\big)\cdot\boldsymbol{v}(x)\,dx \;=\; -\int_\Omega \boldsymbol{u}(x)\,\ldots\,dx + \int_{\partial\Omega} \boldsymbol{u}(x)\,\ldots\,ds$$

03 Consider the equilibrium conditions

$$-\operatorname{div}\boldsymbol{\sigma} \;=\; \boldsymbol{f} \qquad \text{in } \Omega$$

and Hooke's law

$$\boldsymbol{\sigma} \;=\; \lambda \operatorname{div}\boldsymbol{u}\, I + 2\,\mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}),$$

with the linearized strain $\boldsymbol{\varepsilon}(\boldsymbol{u}) \;=\; \frac{1}{2}(\operatorname{grad}\boldsymbol{u} + (\operatorname{grad}\boldsymbol{u})^\top)$. From these identities, derive the Lamé equations

$$-\mu\,\Delta\boldsymbol{u} - (\lambda+\mu)\operatorname{grad}(\operatorname{div}\boldsymbol{u}) \;=\; \boldsymbol{f} \qquad \text{in } \Omega.$$

04 Multiply the Lamé equations by a test function $\boldsymbol{v}$ and integrate over the domain Ω. Then eliminate second-order derivatives using integration by parts and put the result in to the form

$$\int_\Omega \ldots \, dx \; - \; \int_{\partial\Omega} \ldots \, dx \; = \; \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx.$$

Check whether the domain integral coincides with the bilinear form

$$a(u,v) \; = \; \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v}) \, dx \; = \; \int_\Omega \big[\lambda \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v} + 2\,\mu\, \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\big] \, dx$$

and if the integral over the boundary coincides with the expression

$$\int_{\partial\Omega} \boldsymbol{\sigma n} \cdot \boldsymbol{v} \, ds \; = \; \int_{\partial\Omega} \big[\lambda \, (\operatorname{div} \boldsymbol{u}) \, (\boldsymbol{v} \cdot \boldsymbol{n}) + 2\,\mu\, \boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{n} \cdot \boldsymbol{v}\big] \, ds.$$

05 Consider the 2^{nd} boundary value problem for the biharmonic equation:
Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} \Delta^2 u &= f && \text{in } \Omega, \\ u &= g_0 && \text{on } \partial\Omega, \\ \Delta u &= g_2 && \text{on } \partial\Omega, \end{aligned}$$

with given functions f, g_0, $g_2 : \Omega \to \mathbb{R}$. Derive the variational formulation, i.e., specify $a(\cdot,\cdot)$, F, V_g, and V_0. Are there solvability conditions like in the 2^{nd} boundary value problem for Laplace's equation?

06 Consider the domain $\Omega = \{x \in \mathbb{R}^3 : x_1 > 0,\ x_2 > 0,\ x_3 > 0,\ |x| < 1\}$ and the functions

$$\varphi(x) = r^\alpha \quad \text{with} \quad r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

and

$$\boldsymbol{v} = \operatorname{grad} \varphi.$$

(a) Show that $\operatorname{curl} \boldsymbol{v} = 0$.

(b) Find an exponent $\alpha \in \mathbb{R}$ such that

$$\int_\Omega |\boldsymbol{v}|^2 \, dx < \infty \qquad \text{and} \qquad \int_\Omega |\operatorname{div} \boldsymbol{v}|^2 \, dx = \infty,$$

which implies that $\boldsymbol{v} \in H(\operatorname{curl}, \Omega)$ but $\boldsymbol{v} \notin H^1(\Omega)^3)$.
Hint: Use spherical coordinates.