T U T O R I A L

"Computational Electromagnetics"

to the lecture

"Numerical Methods in Electrical Engineering"

Tutorial 02 Thursday, April 22, 2010 (Time: 15:30 – 16:15; Room: T 212)

2D magnetostatic problems lead to the solution of a non-linear boundary value problem of the form (see lectures): Find the third component $u(x) = A_3(x)$ of the vector potential A such that the non-linear potential equations

$$-\mathrm{div}(\nu(x, |\nabla u(x)|)\nabla u(x)) = J_3(x) + \left(\frac{\partial H_{02}}{\partial x_1}(x) - \frac{\partial H_{01}}{\partial x_2}(x)\right),\ x = (x_1, x_2) \in \Omega \quad (1.1)$$

holds together with appropriate boundary conditions, e.g. homogenous Dirichlet boundary conditions

$$u(x) = 0,\ x = (x_1, x_2) \in \Gamma_D = \Gamma_B = \Gamma = \partial\Omega, \quad (1.2)$$

where $\Omega \subset \mathbf{R}^2$ is a bounded Lipschitz domain with the boundary $\Gamma = \partial\Omega$.

05 Derive the variational (weak) formulation of the non-linear boundary value problem (1.1) - (1.2), and reformulate the variational problem as a non-linear operator equation of the form $A(u) = F$ in V_0^* !

Hint: Use also integration by parts at right-hand side term arising from the permanent magnetization !

06 Let us first consider the linear case where the reluctivity $\nu = \nu(x)$ is independent of $|\nabla u(x)|$. Formulate appropriate (practically relevant) conditions for $\nu(x)$, $J_3(x)$, $H_{01}(x)$, and $H_{02}(x)$ such that the assumption of the Lax-Migram theorem are fulfilled ! Derive an estimate of the linear functional F appearing at the right-hand side of the variational formulation as well as of the V_0-ellipticity constant μ_1 and the V_0-boundeness constant μ_2 of the corresponding bilinear form $a(.,.)$!

07* If $\nu(\cdot)\cdot : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ is strongly monotone with the monoticity constant $m > 0$, i.e.

$$(\nu(s)s - \nu(t)t)(s - t) \geq m(s - t)^2,\ \forall\, s, t \in \mathbf{R}_0^+ = [0, \infty), \quad (1.3)$$

then the non-linear operator $A(\cdot) : V_0 \to V_0^*$ defined by the weak formulation of (1.1) - (1.2), see also Exercise 05, is strongly monotone with the same monoticity constant $m > 0$, i.e.

$$\langle A(u) - A(v), u - v\rangle \geq m\|u - v\|_{V_0}^2,\ \forall\, u, v \in V_0, \quad (1.4)$$

if we choose the $H^1(\Omega)$ semi-norm as norm in $V_0 = H_0^1(\Omega)$, i.e. $\|\cdot\|_{V_0} = |\cdot|_{H_0^1(\Omega)}$. Here and below we omit the dependence of the reluctivity ν of the spacial variable

$x = (x_1, x_2)$.

Hints: First, you have to show that the mapping $\nu(|\cdot|)\cdot : \mathbf{R}^2 \to \mathbf{R}^2$ is strongly monotone, i.e. show that, for all $p, q \in \mathbf{R}^2$, we have

$$\begin{aligned}(\nu(|p|)p - \nu(|q|)q)(p-q) &= m\,|p-q|^2 + [(\nu(|p|) - m)p - (\nu(|q|) - m)q] \cdot (p-q) = \ldots \\ &\geq \ldots \\ &\geq m\,|p-q|^2.\end{aligned}$$

Then, setting $p := \nabla u$ and $q := \nabla v$, you can easily prove (1.4).

08* If $\nu(\cdot)\cdot : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ is Lipschitz continuous with the Lipschitz constant $L > 0$, i.e.

$$|\nu(s)s - \nu(t)t)| \leq L|s-t|, \ \forall\, s, t \in \mathbf{R}_0^+ = [0, \infty), \tag{1.5}$$

then the non-linear operator $A(\cdot) : V_0 \to V_0^*$ defined by the weak formulation of (1.1) - (1.2), see also Exercise 05, is Lipschitz continuous with the Lipschitz constant $3L$, i.e.

$$\|A(u) - A(v)\|_{V_0^*} \leq 3L\,\|u - v\|_{V_0}, \ \forall\, u, v \in V_0, \tag{1.6}$$

if we again choose the $H^1(\Omega)$ semi-norm as norm in $V_0 = H_0^1(\Omega)$, i.e. $\|\cdot\|_{V_0} = |\cdot|_{H_0^1(\Omega)}$.

Hints: First, you should show that the non-negative function $\nu(\cdot)$ is bounded by the Lipschitz constant L. Second, show that

$$\begin{aligned}|\nu(|p|)p - \nu(|q|)q| &= |\nu(|p|)(p-q) + (\nu(|p|) - \nu(|q|))q| \\ &\leq \ldots \\ &\leq 3L\,|p-q|.\end{aligned}$$

for all $p, q \in \mathbf{R}^2$.Then, setting again $p := \nabla u$ and $q := \nabla v$, you can easily prove (1.6).

The strong monoticity and the Lipschitz continuity of the non-linear operator $A(\cdot)$ together with linearity and continuity of the functional F induced by the right-hand side of (1.1) (see also Exercise 06) ensure existence and uniqueness of a weak solution $u \in V_0$ of the non-linear operator equation $A(u) = F$ due to the Theorem of Zarantonello (= generalization of Lax-Milgram to non-linear problems), see also Section 1.2.1 of the Lectures in Computational Mechanics.