Mixed Variational Problems (MVP)

(MVP) Find $u \in X$ and $\lambda \in \Lambda$:

$$a(u,v) + b(v,\lambda) = \langle F, v\rangle \quad \forall v \in X$$
$$b(u,\mu) - c(\lambda,\mu) = \langle G, \mu\rangle \quad \forall \mu \in \Lambda$$

$$Au + B^T\lambda = F$$
$$Bu - C\lambda = G$$

Theorem B.4 (Brezzi, 1974)

__Ass.:__ (B.4.1) $F \in X^*$, $G \in \Lambda^*$

(B4.2) $|a(u,v)| \le \alpha_2 \|u\|_X \|v\|_X \quad \forall u,v \in X$

$|b(u,\lambda)| \le \beta_2 \|u\|_X \|\lambda\|_\Lambda \quad \forall u \in X \; \forall \lambda \in \Lambda$

(B4.3) LBB-condition:

$$\sup_{v \in X} \frac{b(v,\lambda)}{\|v\|_X} \ge \beta_1 \|\lambda\|_\Lambda \quad \forall \lambda \in \Lambda$$

(B4.4) Ker B-ellipticity of $a(\cdot,\cdot)$:

$$a(v,v) \ge \alpha_1 \|v\|_X^2 \quad \forall v \in \operatorname{Ker} B$$

__St.:__ $\exists!\, (u,\lambda) \in X \times \Lambda$: (MVP)

$$\|u\|_X \le \frac{1}{\alpha_1}\|F\|_{X^*} + \frac{1}{\beta_1}\left(1+\frac{\alpha_2}{\alpha_1}\right)\|G\|_{\Lambda^*}$$

$$\|\lambda\|_\Lambda \le \frac{1}{\beta_1}\left(1+\frac{\alpha_2}{\alpha_1}\right)\|F\|_{X^*} + \frac{\alpha_2}{\beta_1^2}\left(1+\frac{\alpha_2}{\alpha_1}\right)\|G\|_{\Lambda^*}$$

__Proof__ see Lecture on CM!