Babuška-Aziz-Theorem (1972)

Find $u \in \mathcal{U}$: $a(u,v) = \langle F, v \rangle \;\; \forall v \in \mathcal{V}$

$\Updownarrow$ $\quad A : \mathcal{U} \to \mathcal{V}^* : \; \langle Au, v \rangle := a(u,v) \;\; \forall u \in \mathcal{U} \;\; \forall v \in \mathcal{V}$

$A^* : \mathcal{V} \to \mathcal{U}^*$; $A^* \in L(\mathcal{V}, \mathcal{U}^*)$, $A \in L(\mathcal{U}, \mathcal{V}^*)$

Find $u \in \mathcal{U}$: $\;\; Au = F$ in $\mathcal{V}^*$

Theorem B3 (Babuška-Aziz, 1972)

Let $\mathcal{U}, \mathcal{V}$ - be Hilbert-spaces.
The the linear mapping $A : \mathcal{U} \to \mathcal{V}^*$ is an isomorphism (A = bij., A = cont., A^{-1} = cont.)

iff

the corresponding bilinear form $a(\cdot,\cdot) = \langle A\cdot, \cdot \rangle : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$ satisfies the following conditions:

(B3.1) Continuity, i.e. $\exists \mu_2 = \text{const} > 0$:
$$|a(u,v)| \le \mu_2 \|u\|_{\mathcal{U}} \|v\|_{\mathcal{V}} \quad \forall u \in \mathcal{U} \;\; \forall v \in \mathcal{V};$$

(B3.2) inf-sup-condition, i.e. $\exists \mu_1 = \text{const} > 0$:
$$\inf_{u \in \mathcal{U}} \sup_{v \in \mathcal{V}} \frac{a(u,v)}{\|u\|_{\mathcal{U}} \|v\|_{\mathcal{V}}} \ge \mu_1 = \text{const} > 0;$$

(B3.3) $\forall v \in \mathcal{V} \setminus \{0\} \;\; \exists u \in \mathcal{U}$: $a(u,v) \neq 0$.

<u>Proof</u> see Lecture and Tutorial on CM!