APPENDIX B

Theorem B1: Lax-Milgram (1954)

Ass.: $V, \|\cdot\|_V, (\cdot,\cdot)_V$ – Hilbert space

1. $F \in V^*$ – dual Hilbert space
2. $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ ~ bilinear form:

2a) V-elliptic: $\mu_1 \|v\|_V^2 \leq a(v,v) \quad \forall v \in V$

2b) V-bounded: $|a(u,v)| \leq \mu_2 \|u\|_V \|v\|_V \quad \forall u,v \in V$

St.: $\exists!\, u \in V : a(u,v) = \langle F, v\rangle \quad \forall v \in V$

Linear $\qquad Au = F \text{ in } V^*$

Proof: $Au = F \iff u = u - \tau J_V^{-1}(Au - F) \text{ in } V$

Th. A2 $\qquad \uparrow k \to \infty$ Banach's fixed point theorem

$u_{k+1} = u_k - \tau J_V^{-1}(Au_k - F) \text{ in } V$

Theorem B2: Zarantonello (1960)

Let $V, \|\cdot\|_V, (\cdot,\cdot)_V$ be a Hilbert space, 1) $F \in V^*$, and $A : V \to V^*$ a non-Linear operator satisfying the following conditions:

2a) A is strongly monotone:

$\langle A(u) - A(v), u - v\rangle \geq \mu_1 \|u-v\|_V^2 \quad \forall u,v \in V.$

2b) A is Lipschitz continuous:

$\|A(u) - A(v)\|_{V^*} \leq \mu_2 \|u-v\|_V \quad \forall u,v \in V.$

Then the operator equation

$$A(u) = F$$

non-linear

has a unique determined solution $u \in V$.

Proof: $A(u) = F \iff u = u - \tau J_V^{-1}(A(u) - F)$

Th. A3 $\qquad \uparrow k \to \infty$ Banach's fixed point theorem

$u_{k+1} = u_k - \tau J_V^{-1}(A(u_k) - F)$