APPENDIX A
BASIC THEOREMS

Theorem A1: Riesz' representation theorem

$$X,\ \|\cdot\|_X,\ (u,v)_X = \langle J_X u, v\rangle = \langle J_X u, J_X^{-1} J_X v\rangle =: (J_X u, J_X v)_{X^*}$$

$$J_X \downarrow_{\text{iso}} \uparrow J_X^{-1} \qquad \langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{X^*\times X} : X^* \times X \longrightarrow \mathbb{R}$$

$$X^*,\ \|\cdot\|_{X^*},\ (f,g)_{X^*} = \langle f, J_X^{-1} g\rangle = (J_X^{-1} f, J_X^{-1} g)_X$$

$$\|u\|_X = \|J_X u\|_{X^*},\quad \|J_X\|_{L(X,X^*)} = \|J_X^{-1}\|_{L(X^*,X)} = 1$$

Theorem A2: Banach's fixed point theorem

$X,\ \|\cdot\|_X$ – Banach space

Linear

$\Phi \in L(X,X) : \ \|\Phi\| \le q < 1$

$$\exists!\, x \in X : \ x = \Phi x \xleftarrow[k\to\infty]{} x_{k+1} = \Phi x_k$$

$$\|x - x_k\|_X \le q^k \|x - x_0\|_X$$

$$\|x - x_k\|_X \le \frac{q^k}{1-q} \|x_1 - x_0\|_X$$

$$\|x - x_k\|_X \le \frac{q}{1-q} \|x_k - x_{k-1}\|_X$$

Theorem A3: Banach's fixed point theorem

$X,\ \|\cdot\|_X$ – Banach space

non-linear

$\Phi : D(X) \subseteq X \longrightarrow X$ – non-linear operator

$\emptyset \neq M = \overline{M} \subseteq D(X) \subseteq X$

$\Phi(M) \subset M$

$\|\Phi(x) - \Phi(y)\|_X \le q\, \|x-y\|_X \ \ \forall x,y \in M$ with $0 \le q < 1$

Then $\exists!\, x \in M : \ x = \Phi(x) \xleftarrow[k\to\infty]{} x_{k+1} = \Phi(x_k)$

$$\|x - x_k\|_X \le q^k \|x - x_0\|_X \ \text{etc (1)}$$