Commuting diagram:

$$\begin{tikzcd}
H^1 \cap C^1 \arrow[r, "\nabla"] \arrow[d, "I_h^V"] & H(\mathrm{curl}) \cap C^0 \arrow[r, "\mathrm{curl}"] \arrow[d, "I_h^E"] & L_2 \arrow[d, "I_h^T"] \\
W_h \arrow[r, "\nabla"] & V_h \arrow[r, "\mathrm{curl}"] & S_h
\end{tikzcd}$$

Th. 3.14 ↑ $\boxed{\nabla I_h^V w = I_h^E \nabla w}$

Th. 3.15 ↑ $\boxed{\mathrm{curl}\, I_h^E u = I_h^T \mathrm{curl}\, u}$

where I_h^V – vertex interpolation operator (↑),

I_h^E – edge interpolation operator (↑),

I_h^T – element interpolation operator onto piesewise constant elements with the functional (dof) $\psi_T(s) = \int_T s\,dx$

- **Theorem 3.14:** The commuting identity

(14) $\quad \nabla I_h^V w = I_h^E \nabla w$

holds for all $w \in H^1 \cap C^1$.

Proof: 1) $\nabla I_h^V w \in \bar{V}_h$ due to Theorem 3.9.

2) $I_h^E \nabla w \in \bar{V}_h$ (trivial!)

3) It remains to show that $\psi_{E_{ij}}(\nabla I_h^V w) \overset{!}{=} \psi_{E_{ij}}(I_h^E \nabla w)$!

3a) $\psi_{E_{ij}}(\nabla I_h^V w) = \int_{E_{ij}} \nabla I_h^V w \cdot t\, ds = (I_h^V w)(\bar{V}_j) - (I_h^V w)(\bar{V}_i)$

$= w(\bar{V}_j) - w(\bar{V}_i)$ ✓

3b) $\psi_{E_{ij}}(I_h^E \nabla w) = \psi_{E_{ij}}\Big(\sum_{E_{kl}} \psi_{E_{kl}}(\nabla w)\, \varphi_{E_{kl}}\Big)$

$\psi_{E_{ij}}(\varphi_{E_{kl}}) = \delta_{ij,kl}$

$= \psi_{E_{ij}}(\nabla w) = \int_{E_{ij}} \nabla w \cdot t\, ds = w(\bar{V}_j) - w(\bar{V}_i)$

q.e.d.