Theorem 2.20:

Assume that the solution $u \in H_0(\mathrm{curl}) = V_0$:

(32) $\int_\Omega \nu \,\mathrm{curl}\, u \,\mathrm{curl}\, v + \int_\Omega \varkappa u v \, dx = \int_\Omega f \cdot v \, dx \;\; \forall v \in V_0$

(A1) $0 < \nu_1 \le \nu(x) \le \nu_2 \;\; \forall x \in \Omega$

(A2) $\varkappa \in \mathbb{R}^+$

(A3) $\mathrm{div}\, f = 0$

satisfy the stability estimate

(33) $\|\mathrm{curl}\, u\|_0 + \|u\|_0 \le c \|f\|_0 .$

Assume s-regularity due to Def. 2.18. Then beside (33) there also holds the a priori estimate

(34) $\|u\|_{H^s(\mathrm{curl})} := \left\{ \|\mathrm{curl}\, u\|^2_{H^s} + \|u\|^2_{H^s} \right\}^{1/2} \le c \|f\|_0$

PROOF:

- Testing (32) with $v = \nabla \varphi$; $\varphi \in H_0^1(\Omega)$ gives

$$\varkappa \int_\Omega u \cdot \nabla \varphi \, dx = 0, \quad \text{i.e. } \mathrm{div}\, u = 0.$$

Thus, $u \in H_0(\mathrm{curl}) \cap H(\mathrm{div})$!

- Lemma 2.19: $u \in H^s(\Omega)$ and

$$\|u\|_{H^s(\Omega)} \le c \left(\|\mathrm{curl}\, u\|_0 + \underbrace{\|\mathrm{div}\, u\|_0}_{0} \right) \overset{(33)}{\le} \tilde{c} \|f\|_0 .$$

- Now, set $B = \mathrm{curl}\, u \in H_0(\mathrm{div})$: $\mathrm{div}\, B = 0$.
- Furthermore, from

$$(B, \mathrm{curl}\, v)_0 + \varkappa (u, v)_0 = (f, v)_0 ,$$

we get

$$\mathrm{curl}\, B = f - \varkappa u \in L_2$$

Again, Lemma 2.19 gives

$$\|B\|_{H^s} \le c \left\{ \|\mathrm{curl}\, B\|_0 + \|\mathrm{div}\, B\|_0 \right\} = c \|f - \varkappa u\| \le c \|f\|_0$$