- **PROOF of Lemma 2.19:**
- We prove the case $u \in H_0(\mathrm{curl}) \cap H(\mathrm{div})$; the other one is similar.
According to Theorem 2.19, there exists a regular decomposition

$$u = \nabla\varphi + z$$

with $\varphi \in H_0^1(\Omega)$ and $z \in [H_0^1(\Omega)]^3$: $\|z\|_{H^1} \le c\|\mathrm{curl}\, u\|_{L_2}$.
- $\varphi \in H_0^1(\Omega)$ obviously satisfies the variational eqn.:

$$(\nabla\varphi, \nabla\psi)_0 = (u - z, \nabla\psi)_0 \quad \forall \psi \in H_0^1(\Omega)$$

i.e. the Dirichlet problem for $-\Delta\varphi = -\mathrm{div}(u-z)$.
- The rhs $-\mathrm{div}(u-z) \in L_2(\Omega)$ since $z \in [H_0^1(\Omega)]^3$ and $u \in H(\mathrm{div})$!
- s-regularity $\Rightarrow$ $\varphi \in H^{1+s}(\Omega)$ and $\|\varphi\|_{H^{1+s}} \le c\|\mathrm{rhs}\|_{L_2}$
$\Rightarrow$ $\nabla\varphi \in H^s(\Omega)^3$ and $\|\nabla\varphi\|_{H^s(\Omega)} \le c\,\|\mathrm{div}(u-z)\|_{L_2}$

$$\lesssim \|\mathrm{div}\, z\|_0 + \|\mathrm{div}\, u\|_0$$

$$\lesssim \|z\|_1 + \|\mathrm{div}\, u\|_0$$

$$\lesssim \|\mathrm{curl}\, u\|_0 + \|\mathrm{div}\, u\|_0$$

- Therefore, we have

$$\|u\|_{H^s} \le \|\nabla\varphi\|_{H^s} + \|z\|_{H^s} \lesssim$$

$\lesssim \|\mathrm{curl}\, u\|_0$

$$\lesssim \|\mathrm{curl}\, u\|_0 + \|\mathrm{div}\, u\|_0 + \|z\|_1$$

$$\lesssim \|\mathrm{curl}\, u\|_0 + \|\mathrm{div}\, u\|_0,$$

where $\|\cdot\|_0 = \|\cdot\|_{L_2(\Omega)}$ and $\|\cdot\|_1 = \|\cdot\|_{H^1(\Omega)}$. q.e.d.