Theorem 2.12: (Regular Decomposition)

a) Let $u \in H(\mathrm{curl})$. Then there exists a decomposition

$$u = \nabla \varphi + z,$$

with $\varphi \in H^1(\Omega)$ and $z \in [H^1(\Omega)]^3$ such that

$\|\varphi\|_{H^1} \leq c \|u\|_{H(\mathrm{curl})}$ and $\|z\|_{H^1} \leq c \|\mathrm{curl}\, u\|_{L_2}$.

b) If $u \in H_0(\mathrm{curl})$, then there exists a decomposition with $\varphi \in H_0^1(\Omega)$ and $z \in [H_0^1(\Omega)]^3$.

Proof: a) Let $u \in H(\mathrm{curl})$ be given.

- Then $q = \mathrm{curl}\, u$ satisfies $\mathrm{div}\, q = 0$.
- Lemma 2.10: $\exists\, z \in [H^1]^3$: $q = \mathrm{curl}\, z = \mathrm{curl}\, u$

  $\|z\|_{H^1} \leq c \|q\|_{L_2} = c \|\mathrm{curl}\, u\|_{L_2}$
- $\mathrm{curl}(u - z) = 0$, i.e. $\exists\, \varphi \in H^1$: $u - z = \nabla \varphi$ (de Rham)
- We choose φ such that $(\varphi, 1) = \int_\Omega \varphi\, dx = 0$.

  Then the bounds for the norms follow from

$$\|\varphi\|_{H^1(\Omega)} \lesssim \|\nabla \varphi\|_{L_2} \leq \|u\|_{L_2} + \|z\|_{L_2}$$

$$\leq \|u\|_{L_2} + \|z\|_{H^1} \lesssim \|u\|_{H(\mathrm{curl})}.$$

b) Let $u \in H_0(\mathrm{curl}) = H_0(\mathrm{curl}, \Omega)$ be given.

.... (mms)