- <u>Theorem 2.11:</u> (Helmholtz decomposition)

Let $q \in [L_2(\Omega)]^3$. Then there exists a decomposition

$$q = \nabla \varphi + \operatorname{curl} z,$$

where the following choices for φ and z are possible:

(i) $\varphi \in H^1 \wedge z \in [H^1]^3 : \operatorname{div} z = 0$

(ii) $\varphi \in H^1 \wedge z \in [H^1_0]^3$

(iii) $\varphi \in H^1 \wedge z \in H_0(\operatorname{curl}) : \operatorname{div} z = 0$

(iv) $\varphi \in H^1_0 \wedge z \in [H^1]^3 : \operatorname{div} z = 0$

(v) $\varphi \in H^1_0 \wedge z \in H(\operatorname{curl}) : \operatorname{div} z = 0 \wedge tr_n z = 0$

The corresponding norms are bounded by $\|q\|_{L_2(\Omega)}$.

<u>Proof:</u> $q \in [L_2(\Omega)]^3$ given

(i)-(iii)
- $\varphi \in H^1(\Omega)|_{\mathbb{R}}$ as the weak solution of the Neumann problem:

$$(\nabla\varphi, \nabla v)_{L_2(\Omega)} = (q, \nabla v)_{L_2(\Omega)} \quad \forall v \in H^1|_{\mathbb{R}}$$

- Then: $q - \nabla\varphi \in H(\operatorname{div}) : \operatorname{div}(q - \nabla\varphi) = 0 \wedge tr_n(q - \nabla\varphi) = 0$
- Lemma 2.9 $\Rightarrow \exists\, z \in *$: $q - \nabla\varphi = \operatorname{curl} z$

$*$ (i)-(iii)

(iv)+(v)
- $\varphi \in H^1_0(\Omega)$ as the weak solution of the Dirichlet problem:

$$(\nabla\varphi, \nabla v) = (q, \nabla v) \quad \forall v \in H^1_0(\Omega)$$

- Then: $q - \nabla\varphi \in H(\operatorname{div}) : \operatorname{div}(q - \nabla\varphi) = 0$
- Lemma 2.10 $\Rightarrow \exists\, z \in *$: $q - \nabla\varphi = \operatorname{curl} z$

$*$ (iv)-(v)

q.e.d.