2.1.3. Helmholtz Decomposition: $u = \nabla\varphi + \operatorname{curl} z$

- **Lemma 2.9:** $H(\operatorname{div}) = H(\operatorname{div}, \Omega), \ldots$

  Assume that $q \in H(\operatorname{div}): \operatorname{div} q = 0$ and $tr_n q = 0$.
  Then there exists a vector function z (↓) such that

  $$q = \operatorname{curl} z,$$

  where z can be chosen such that

  (i) $z \in [H^1(\Omega)]^3: \operatorname{div} z = 0 \wedge |z|_{H^1(\Omega)} \leq \|q\|_{L_2(\Omega)}$,

  OR (ii) $z \in [H^1_0(\Omega)]^3: \|z\|_{H^1(\Omega)} \leq c\,\|q\|_{L_2(\Omega)}$

  OR (iii) $z \in H_0(\operatorname{curl}): \operatorname{div} z = 0 \wedge \|z\|_{H(\operatorname{curl})} \leq \|q\|_{L_2(\Omega)}$

  Proof: via Fourier transformation!

- **Lemma 2.10:**

  Assume that $q \in H(\operatorname{div}): \operatorname{div} q = 0$.
  Then there exists a vector function z (↓) such that

  $$q = \operatorname{curl} z,$$

  where z can be chosen in such a way that

  (i) $z \in [H^1(\Omega)]^3: \operatorname{div} z = 0 \wedge |z|_{H^1(\Omega)} \leq \|q\|_{L_2(\Omega)}$

  OR (ii) $z \in H(\operatorname{curl}): \operatorname{div} z = 0 \wedge tr_n z = 0 \wedge \|z\|_{H(\operatorname{curl})} \leq \|q\|_{L_2(\Omega)}$.