- **Dual Space:** $H^{-1/2}(\Gamma) = (H^{1/2}(\Gamma))^*$

  Duality Product: $\langle \cdot , \cdot \rangle_{H^{-1/2} \times H^{1/2}} : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to \mathbb{R}$

  Remark:

  $$\langle j_s, v \rangle = \int_\Gamma j_s \cdot v \, ds \le \|j_s\|_{H^{-1/2}(\Gamma)} \|v\|_{H^{1/2}} \le \|j_s\|_{H^{-1/2}} \cdot c_T \|v\|_{H^1(\Omega)}$$

  $\uparrow$ $j_s \in L_2(\Gamma) \subset H^{-1/2}(\Gamma)$ $\quad \forall v \in H^1(\Omega)$

- **Integration by parts (IbyP):**

  $$(5) \quad \int_\Omega \nabla u \cdot \varphi \, dx + \int_\Omega u \cdot \operatorname{div} \varphi \, dx = \int_\Gamma tr_\Gamma u \; \varphi \cdot n \, ds$$

  $$\forall u \in H^1(\Omega) \ \forall \varphi \in (C^\infty(\overline{\Omega}))^3$$

- $\overset{\circ}{H}{}^1(\Omega) := \{ v \in H^1(\Omega) : tr_\Gamma u = 0 \} = \overline{\overset{\circ}{C}{}^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}$

- <u>Lemma 2.3:</u>

  Let $\Omega_1, \dots, \Omega_m$ be a non-overlapping domain decomposition (DD) of Ω, i.e. $\overline{\Omega} = \bigcup \overline{\Omega}_i$ and $\Omega_i \cap \Omega_j = \emptyset \ \forall i \ne j$. Let $u_i \in H^1(\Omega), i = \overline{1,m}$:

  $tr_{\Gamma_{ij}} u_i = tr_{\Gamma_{ij}} u_j \ \forall \Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j : meas_{d-1} \Gamma_{ij} > 0.$

  Then the piecewise defined function

  $u := \{ u|_{\Omega_i} = u_i, \ i = \overline{1,m} \} \in H^1(\Omega)$ and $(\nabla u)|_{\Omega_i} = \nabla u_i \ \forall i.$

  <u>Proof:</u> Let $g_i := \nabla u_i$ be the local weak gradients. Set $g := g_i$ on $\Omega_i \ \forall i = \overline{1,m}$, i.e. $g \in L_2(\Omega)$. Then $\forall \varphi \in (\overset{\circ}{C}{}^\infty(\Omega))^3$, we have

  $$-\int_\Omega u \operatorname{div} \varphi \, dx = \sum_{i=1}^m \int_{\Omega_i} u_i \operatorname{div} \varphi \, dx \overset{(5)}{=}$$

  $$= \sum_i \left[\int_{\Omega_i} \nabla u_i \cdot \varphi \, dx - \int_{\partial\Omega_i} tr_{\partial\Omega_i} u_i \; \varphi \cdot n_i \, ds \right]$$

  $$= \sum_i \int_{\Omega_i} g_i \cdot \varphi \, dx - \sum_{\Gamma_{ij}} \int_{\Gamma_{ij}} \underbrace{(tr_{\Gamma_{ij}} u_i - tr_{\Gamma_{ij}} u_j)}_{=0} \varphi \cdot n_i \, ds$$

  $$= \sum_i \int_{\Omega_i} g_i \cdot \varphi \, dx = \int_\Omega g \cdot \varphi \, dx, \text{ i.e. } g = \nabla u \text{ (Def. 2.1)!}$$

  q.e.d.