59 The definitions of consistency, stability, and convergence depend on the underlying norms. In the following we use $\|\cdot\|_{X_\tau}$ instead of $\|\cdot\|_{Y_\tau}$. Using Exercise **53** and following your lecture notes, show the estimate

$$\|e_\tau\|_{X_\tau} \;\leq\; C\,\|\psi_\tau\|_{X_\tau}\,.$$

Furthermore, show that if the exact solution fulfills $u \in C^2([0,\,T],\mathbb{R}^n)$, then

$$\|\psi_\tau\|_{X_\tau} \;\leq\; K\,\tau\,,$$

and conclude a corresponding estimate for the global error.
Hint: $K = \max_{s\in[0,\,T]}\|u''(s)\| < \infty$.

60 Consider the 2-stage explicit Runge-Kutta method

$$\begin{aligned} g_1 &= u_j\,,\\ g_2 &= u_j + \tau_j\,a_{21}\,f(t,\,g_1),\\ u_{j+1} &= u_j + \tau_j\left[b_1\,f(t_j,\,g_1) + b_2\,f(t_j + c_2\,\tau_j,\,g_2)\right], \end{aligned}$$

for the solution of the initial value problem to find $u : [0,\,T] \to \mathbb{R}$ such that

$$\begin{aligned} u'(t) &= f(t,\,u(t)) \qquad \forall t \in (0,\,T),\\ u(0) &= u_0\,, \end{aligned}$$

for given $f : [0,\,T] \times \mathbb{R} \to \mathbb{R}$ sufficiently smooth, and $u_0 \in \mathbb{R}$. Provide a Taylor series expansion of the *local error* of the form

$$d(t+\tau) \;=\; A_0 + \tau\,A_1 + \tau^2\,A_2 + \tau^3\,A_3 + \mathcal{O}(\tau^4),$$

with the expressions A_1, A_2, and A_3 only depending on a_{21}, b_1, b_2, c_2, f and its derivatives, but not on τ.

61 Continue exercise **60** and find necessary and sufficient conditions on the coefficients a_{21}, b_1, b_2, and c_2 such that the consistency order of the method is at least 2, i. e., that for all sufficiently smooth functions f, we have

$$A_0 \;=\; A_1 \;=\; A_2 \;=\; 0.$$

62 Consider the classical Runge-Kutta method of order 4,

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

Show that this method has the stability function

$$R(z) \;=\; 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4\,,$$

and that $e^z - R(z) = \mathcal{O}(z^5)$ as $z \to 0$.

Programming (in C, C^{++}, or `matlab`)

Consider the problem to find $u : [0,\, T] \to \mathbb{R}$ such that

$$\begin{aligned} u'(t) &= -50\, u(t) \qquad \forall t \in (0,\, T), \\ u(0) &= 1, \end{aligned}$$

with $T = 1$. Note that the exact solution is given by $u(t) = e^{-50\,t}$.

63 Implement the *explicit* Euler method for the above problem with fixed time steps. Run it for the choices $\tau = 1/60$, $1/30$, $1/26$, $1/24$, and $1/20$. For each run, plot the exact and numerical solution.

64 Implement the *implicit* Euler method for the above problem with fixed time steps. Run it for the same choices of τ as in exercise 63. For each run, plot the exact and numerical solution.