53 Consider the sequence $u_\tau \in X_\tau$ defined according to the explicit Euler method and the perturbed sequence $v_\tau \in Y_\tau$ defined by

$$\begin{aligned} u_{j+1} &= u_j + \tau_j\, f(t_j,\, u_j), \\ v_{j+1} &= v_j + \tau_j \left[f(t_j,\, v_j) + y_{j+1} \right], \end{aligned}$$

with $v_0 = u_0 + y_0$, for given initial data u_0 and given perturbations $y_\tau \in Y_\tau$. Assume further, that f satisfies the Lipschitz condition

$$\| f(t,\, v) - f(t,\, w) \| \;\leq\; L\, \| v - w \| \qquad \forall v,\, w \in \mathbb{R}^n \quad \forall t \in [0,\, T].$$

Show that

$$\| v_j - u_j \| \;\leq\; e^{(t_j - t_0)L}\, \| y_0 \| + \frac{1}{L} \left(e^{(t_j - t_0)L} - 1 \right) \max_{k=1,\dots,j} \| y_k \|.$$

Hint: Follow your lecture notes and use that $e^{(t_j - t_k)L}\, \tau_{k-1} \leq \int_{t_{k-1}}^{t_k} e^{(t_j - s)L} ds$ (and show this!).

Consider the general initial value problem to find $u : [0,\, T] \to X$ such that

$$\begin{aligned} u'(t) &= f(t,\, u(t)) \qquad \forall t \in \mathbb{R}_0^+ , \\ u(0) &= u_0 , \end{aligned} \tag{10.1}$$

with $f : \mathbb{R}_0^+ \times X \to X$ and $u_0 \in X$, where X is a Banach space.

54 Assume that there exists a constant $L > 0$ such that

$$\| f(t,\, w) - f(t,\, v) \| \;\leq\; L\, \| w - v \| \qquad \forall t \in \mathbb{R}_0^+ \quad \forall v,\, w \in X, \tag{10.2}$$

where $\| \cdot \|$ is a norm in X. Show that for each given $t_j > 0$ and $u_j \in X$, there exists a unique solution $u_{j+1} \in X$ to the implicit equation

$$u_{j+1} \;=\; u_j + \tau_j\, f(t_{j+1},\, u_{j+1}),$$

if $\tau < 1/L$. *Hint:* Use Banach's fixed point theorem.

55 Assume that X is a Hilbert space with the inner product $(\cdot,\, \cdot)$, and that

$$\big(f(t,\, w) - f(t,\, v),\, w - v \big) \;\leq\; 0 \qquad \forall t \in \mathbb{R}_0^+ \quad \forall v,\, w \in X, \tag{10.3}$$

holds additionally to (10.2). Show that for each given $\tau > 0$, $t_j > 0$, and $u_0 \in X$, there exists a unique solution $u_{j+1} \in X$ to the implicit equation

$$u_{j+1} \;=\; u_j + \tau_j\, f(t_{j+1},\, u_{j+1}).$$

Hint: Apply Banach's fixed point theorem to the equivalent equation

$$u_{j+1} \;=\; G(u_{j+1}) := (1 - \rho) u_{j+1} + \rho \big[u_j + \tau_j\, f(t_{j+1},\, u_{j+1}) \big],$$

for some parameter $\rho \in (0,\, 1)$, which you should choose such that G is a contraction.

For the following exercises we consider the implicit Euler method:

$$u_{j+1} \;=\; u_j + \tau\, f(t_{j+1},\, u_{j+1}),$$

here with equidistant time steps $\tau_j = \tau$. Let

$$\psi_\tau(t+\tau) := \frac{1}{\tau}\big[u(t+\tau) - u(t)\big] - f(t+\tau,\, u(t+\tau))$$

denote the *consistency error* of the implicit Euler method, where $u(t)$ is the exact solution to problem (10.1). Furthermore, let $e_k := u(t_k) - u_k$ denote the *global error.*

$\boxed{56}$ Show that the following estimates holds, provided $u \in C^2(\mathbb{R}_0^+)$:

$$\|\psi(t+\tau)\| \;\le\; \int_t^{t+\tau} \|u''(\sigma)\|\, d\sigma.$$

$\boxed{57}$ Obviously, it follows from the above definitions that

$$\begin{aligned} u(t_{j+1}) \;&=\; u(t_j) + \tau\, f(t_{j+1},\, u(t_{j+1})) + \tau\, \psi_\tau(t_{j+1}), \\ e_{j+1} \;&=\; e_j + \tau\, \big[f(t_{j+1},\, u(t_{j+1})) - f(t_{j+1},\, u_{j+1})\big] + \psi_\tau(t_{j+1}). \end{aligned} \tag{10.4}$$

Let the assumptions of exercise $\boxed{55}$ be fulfilled. Show that

$$\|e_{j+1}\| \;\le\; \|e_j\| + \tau\, \|\psi_\tau(t_{j+1})\|.$$

Hint: Multiply (10.4) by e_{j+1} and apply Cauchy's inequality to the right hand side.

$\boxed{58}$ Let the assumptions of exercise $\boxed{55}$ be fulfilled. Show that

$$\|u(t_j) - u_j\| \;\le\; \tau \int_0^{t_j} \|u''(\sigma)\|\, d\sigma,$$

provided $u \in C^2([0, \tau_j])$.