We consider the abstract problem: find $u \in H^1\big((0,T), V; H\big)$ such that

$$\begin{aligned} \underbrace{\frac{d}{dt}(u(t),\, v)_H}_{=\langle u'(t), v\rangle} + a(u(t),\, v) &= \langle F(t),\, v\rangle \qquad \forall v \in V \quad \forall t \in (0,T) \text{ a.e.}, \\ u(0) &= u_0\,, \end{aligned} \tag{9.1}$$

for given $u_0 \in H$ and $F \in L^2\big((0,T), V^*\big)$, where (V, H, V^*) is an evolution triple, i. e.,

- V and H are separable Hilbert spaces,
- $V \subset H$ and V is dense in H,
- there exists a constant $c > 0$ such that $\|v\|_H \le c\,\|v\|_V$ for all $v \in V$.

47 Show that for all $\lambda \in \mathbb{R}$: the function $u \in H^1\big((0,T), V; H\big)$ is a solution of (9.1) *if and only if* $u_\lambda \in H^1\big((0,T), V; H\big)$ solves

$$\begin{aligned} \frac{d}{dt}(u_\lambda(t),\, v)_H + a_\lambda(u_\lambda(t),\, v) &= \langle F_\lambda(t),\, v\rangle \qquad \forall v \in V \quad \forall t \in (0,T) \text{ a.e.}, \\ u_\lambda(0) &= u_0\,, \end{aligned} \tag{9.2}$$

where

$$u_\lambda(t) = e^{-\lambda t} u(t), \qquad a_\lambda(w,v) = a(w,v) + \lambda (w,v)_H\,, \qquad F_\lambda(t) = e^{-\lambda t} F(t).$$

Hint: use the definition of the weak time derivative $u'(t)$ to compute $u'_\lambda(t)$.

48 Show that Theorem 2.9 from your lecture notes holds also when we replace the assumption of coercivity by the weaker assumption that there exist constants $\lambda \in \mathbb{R}$ and $\mu_1 > 0$ such that

$$a(v,v) + \lambda\,\|v\|_H^2 \;\ge\; \mu_1\,\|v\|_V^2 \qquad \forall v \in V$$

(such an inequality is called *Gårding inequality*).
Hint: use Exercise 47.

49 Consider the bilinear form

$$a(w,\, v) := \int_0^1 a(x)\,\frac{\partial w}{\partial x}(x)\,\frac{\partial v}{\partial x}(x) + b(x)\,\frac{\partial w}{\partial x}(x)\,v(x) + c(x)\,w(x)\,v(x)\,dx$$

in $H^1(0,1)$ with $a, b, c \in L^\infty(0,1)$, where $a_0 := \inf_{x\in(0,1)} a(x) > 0$. Show the Gårding inequality: there exist constants $\lambda \in \mathbb{R}$ and $\mu_1 > 0$ such that

$$a(v,\, v) + \lambda\,\|v\|^2_{L^2(0,1)} \;\ge\; \mu_1\,\|v\|^2_{H^1(0,1)} \qquad \forall v \in H^1(0,1).$$

Hint: Choose λ such that the assumptions of Excercise 08 (Tutorial 2) hold for the bilinear form $a_\lambda(w,\, v) := a(w,\, v) + \lambda\,(w,v)_{L^2(0,1)}$.

50 Let $C^1([0,T],V)$ denote the space of continuous functions in $[0,T]$ with values in the Hilbert space V that have a continuous *classical* derivative, i. e., for $v \in C^1([0,T],V)$ the limit

$$v'(t) := \lim_{\tau \to 0} \frac{1}{\tau}\big(v(t+\tau) - v(t)\big)$$

exists for all $t \in [0,T]$ and the function $v' : [0,T] \to V$ is continous.
Show that for all s, $t \in [0,T]$ and for all $v \in C^1([0,T],V)$:

$$\frac{1}{2}\big(v(t),\, v(t)\big)_H \;=\; \frac{1}{2}\big(v(s),\, v(s)\big)_H + \int_s^t \big(v'(\sigma),\, v(\sigma)\big)_H \, d\sigma. \tag{9.3}$$

Hint: Prove and use the identity

$$\frac{1}{2}\Big[\big(v(\sigma),\, v(\sigma)\big)_H\Big]' \;=\; \big(v'(\sigma),\, v(\sigma)\big)_H \,.$$

51 Prove Lemma 2.7, i. e., show that there exists a constant $C > 0$ with

$$\max_{t\in[0,\,T]} \|v(t)\|_H \;\le\; C\,\|v\|_{H^1((0,T),V;H)} \qquad \forall v \in C^1([0,T],V).$$

Hint: Integrate identity (9.3) with respect to s over $[0,T]$. Note that $\|v'\|^2_{L^2((0,T),V^*)}$ is the integral over the (square of the) V^*-norm of the mapping $w \mapsto (v'(t), w)_H$. Show and use that

$$(v'(\sigma),\, w)_H \;\le\; \|v'(\sigma)\|_{V^*}\,\|w\|_V \;\le\; \frac{1}{2}\Big[\|v'(\sigma)\|^2_{V^*} + \|w\|^2_{V^*}\Big] \qquad \forall w \in V.$$

52 Assume that $a(\cdot,\cdot)$ is bounded and coercive with coercivity constant $\mu_1 > 0$. Show that

$$\|\theta_h(t)\|_H \;\le\; e^{-\mu_1 c^{-2}\,t}\,\|\theta_h(0)\|_H + \int_0^t e^{-\mu_1 c^{-2}(t-s)}\,\|\rho_h'(s)\|_H\, ds,$$

where θ_h and ρ_h are defined according to the lecture notes.
Hint: Bound the term $\frac{d}{dt}\|\theta_h(t)\|_H$ as in the lecture notes but use the coercivity of $a(\cdot,\cdot)$. Multiply the entire estimate by $e^{\mu_1 c^{-2}\,t}$ and investigate the term

$$\frac{d}{dt}\Big[e^{\mu_1 c^{-2}\,t}\,\|\theta_h(t)\|_H\Big]$$

in a side computation.