22 Let $\mathcal{T}_h$ be an equidistant subdivision of (0, 1). Show (analogously to the L^2-estimate the lecture) that there exists a constant $C_1 > 0$ independent of h, such that

$$|v - I_h v|_{H^1(0,1)} \leq C_1\, h\, \|v''\|_{L^2(0,1)} \qquad \forall v \in C^2[0,\, 1]\,. \tag{5.1}$$

23 Show that all expressions in (5.1) are continuous with respect to the H^2-norm.
Hint: Show that the expressions are Lipschitz-continuous.
Then (5.1) follows for all $v \in H^2(0,\,1)$ due to the density of $C^2[0,\,1]$ in that space (closure principle).

24 Let $\mathcal{T}_h$ be an equidistant subdivision of (0, 1), let V_{0h} be the space of continuous piecewise affine linear functions that vanish at 0, and let K_h denote the stiffness matrix corresponding to our model problem. Show that there exists a constant $C_2 > 0$ independent of h such that

$$\kappa(K_h) \;\geq\; C_2\, h^{-2}\,.$$

Hint: Use the Rayleigh quotient for the special vector $\underline{v}_h = (1,\,0,\,\ldots,\,0)^\top$ in order to obtain a lower bound for $\lambda_{\max}(K_h)$. For an upper bound of $\lambda_{\min}(K_h)$ use $\underline{v}_h = (h,\,2h,\,3h,\ldots,1)^\top$.

25 Let $\mathcal{T}_h$ be an equidistant subdivision of (0, 1) and let V_{0h} be the space of continuous piecewise affine linear functions vanising at 0. Let M_h denote the mass matrix for our model problem. Show that there exists a constant $C_3 > 0$ independent of h such that

$$\kappa(M_h) \;\leq\; C_3\,.$$

Programming

26 Write a function `ImplementDirichletBC(↓i, ↓g, ↕matrix, ↕vector)` to implement the Dirichlet boundary condition

$$u(x_i) = g_D(x_i)$$

for a given value `g`$=g_D(x_i)$ at the boundary node x_i identified by the index `i`$=i$. The function `ImplementDirichletBC` must update the stiffness matrix `matrix` and the load vector `vector`, after applying

```
AssembleStiffnessMatrix
AssembleLoadVector
ImplementRobinBC
```

Here, instead of *deleting* rows or columns from the matrix, we stay with the $(n_h+1)\times(n_h+1)$ matrix using the following technique.
Suppose that applying `AssembleStiffnessMatrix`, `AssembleLoadVector` and `ImplementRobinBC` yields the linear system

$$\begin{pmatrix} K_{00} & K_{01} & K_{02} & K_{03} \\ K_{10} & K_{11} & K_{12} & K_{13} \\ K_{20} & K_{21} & K_{22} & K_{23} \\ K_{30} & K_{31} & K_{32} & K_{33} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

and that we want to impose the Dirichlet boundary condition $u_0 = u(x_0) = g_D(x_0) = g_0$. Then, we can replace the first equation by $K_{00}\, u_0 = K_{00}\, g_0$ and substitute u_0 by g_0 in the remaining equations. The modified system reads

$$\begin{pmatrix} K_{00} & 0 & 0 & 0 \\ 0 & K_{11} & K_{12} & K_{13} \\ 0 & K_{21} & K_{22} & K_{23} \\ 0 & K_{31} & K_{32} & K_{33} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} K_{00}\, g_0 \\ f_1 - K_{10}\, g_0 \\ f_2 - K_{20}\, g_0 \\ f_3 - K_{30}\, g_0 \end{pmatrix}.$$

Implement this in an efficient way.

27 Implement an efficient Gauss/Thomas type solver for our system $K_h\, \underline{u}_h = \underline{f}_h$.
Hint 1: Exploit the tridiagonal structure (and maybe also the symmetry).
Hint 2: If you are lazy, take your inspiration from
`http://en.wikipedia.org/wiki/Tridiagonal_matrix_algorithm`

28* (BONUS exercise)

Solve the model problem

$$\begin{aligned} -u''(x) &= f(x) \qquad \forall x \in (0,\,1), \\ u(0) &= g_D \\ u'(1) &= \alpha\left(g_R - u(1)\right), \end{aligned}$$

with $f(x) = 8$, $g_D = -1$, $\alpha = 1$, $g_R = 1$ for different (equidistant) meshes ($h = 1/10$, $h = 1/20$, $h = 1/100$ etc.) and visualize the solution (e. g., using `gnuplot`, `matlab`, etc.).