04 Consider the Dirichlet boundary value problem

$$\begin{aligned} -\big(a(x)\,u'(x)\big)' &= 1 \qquad \forall x \in (0,\,1)\,, \\ u(0) &= 0\,, \qquad a(1)\,u'(1) \;=\; 0\,, \end{aligned} \tag{2.3}$$

where $a(x) = \sqrt{2x - x^2}$. Justify that

$$u(x) = \sqrt{2x - x^2}$$

is a *classical* solution of (2.3), i. e., $u \in X := C^2(0,\,1) \cap C^1(0,\,1] \cap C[0,\,1)$. Furthermore, show that

$$\int_0^1 |u'(x)|^2\,dx = \infty\,.$$

Note: This example shows that $u \notin H^1(0,\,1)$, i. e., u is no *weak* solution.

05 Let the coefficient $a \in L_\infty(0,\,1)$ be defined by

$$a(x) = \begin{cases} a_1 & \text{for } x \in \big[0,\,\frac{1}{2}\big]\,, \\ a_2 & \text{for } x \in \big(\frac{1}{2},\,1\big]\,, \end{cases}$$

with positive constants $a_1 \neq a_2$. Derive a variational formulation for the boundary value problem

$$\begin{aligned} -a(x)\,u''(x) &= f(x) \qquad \forall x \in (0,\,1) \setminus \{\tfrac{1}{2}\}\,, \\ u(0) &= g_1\,, \qquad u(1) \;=\; g_2\,, \end{aligned}$$

with the *transmission conditions*

$$\begin{aligned} u(\tfrac{1}{2}-) &= u(\tfrac{1}{2}+)\,, \\ a_1\,u'(\tfrac{1}{2}-) &= a_2\,u'(\tfrac{1}{2}+)\,, \end{aligned}$$

where, $w(\frac{1}{2}-)$ and $w(\frac{1}{2}+)$ denote the left sided and right sided limit of a function w, respectively. *Hint:* Integration by parts is only valid on the sub-intervals, separately.

06 Derive the variational formulation

$$\text{find } u \in V_g : \quad a(u,\,v) = \langle f,\,v\rangle \qquad \forall v \in V_0 \tag{2.4}$$

of the pure Neumann boundary value problem

$$\begin{aligned} -u''(x) &= f(x) \qquad \text{for } x \in (0,\,1)\,, \\ u'(0) &= g_0\,, \\ -u'(1) &= g_1\,, \end{aligned}$$

and show the following statements:

(a) If (2.4) has a solution, then

$$\langle f,\,c\rangle = 0, \qquad \forall c \in \mathbb{R}\,. \tag{2.5}$$

(b) If u is a solution of (2.4), then, for any constant $c \in \mathbb{R}$, $\widehat{u} := u + c$ is also a solution.

(c) If we choose $c = -\int_0^1 u(x)\,dx$, then

$$\widehat{u} \in \widehat{V} = \{v \in H^1(0,\,1) : \int_0^1 v(x)\,dx = 0\}$$

(d) If $\widehat{u} \in \widehat{V}$ solves the variational problem

$$a(\widehat{u},\,\widehat{v}) = \langle f,\,\widehat{v}\rangle \quad \forall v \in \widehat{V}\,,$$

and if the condition (2.5) holds, then $\widehat{u}$ solves also (2.4).
Hint: Each test function $v \in H^1(0,\,1)$ can be written as $v(x) = \widehat{v}(x) + \overline{v}$ with $\overline{v} = \int_0^1 v(x)\,dx$ and $\widehat{v} \in \widehat{V}$.

In the lecture, the coercivity of the bilinear form

$$a(w,\,v) \;=\; \int_0^1 \big[a(x)\,w'(x)\,v'(x) + b(x)\,w'(x)\,v'(x) + c(x)\,w(x)\,v(x)\,\big]dx\,, \tag{2.6}$$

on the space $V_0 = \{v \in H^1(0,\,1) : v(0) = 0\}$ has been shown for the special case $a(x) = 1$, $b(x) = 0$, $c(x) = 0$. In the following three exercises, we consider more general cases with $a,\,b,\,c \in L_\infty(0,\,1)$. Throughout, you will have to use the estimate

$$a(v,\,v) \;\geq\; a_0\,|v|^2_{H^1(0,\,1)} + \int_0^1 b(x)\,v'(x)\,v(x)\,dx + c_0\,\|v\|^2_{L_2(0,\,1)} \tag{2.7}$$

(which is rather easily shown), where $a_0 = \inf\limits_{x\in(0,\,1)} a(x)$ and $c_0 = \inf\limits_{x\in(0,\,1)} c(x)$.

07 Show the coercivity of $a(w,\,v)$ on $V_0 = \{v \in H^1(0,\,1) : v(0) = 0\}$ under the assumptions

$$a_0 > 0\,, \qquad C_F\,\|b\|_{L_\infty(0,\,1)} < a_0\,, \qquad c_0 \geq 0\,,$$

where C_F is the constant in Friedrichs' inequality.
Hint: Use Cauchy's inequality to show the estimate

$$\int_0^1 b(x)\,v'(x)\,v(x)\,dx \;\geq\; -\|b\|_{L_\infty(0,\,1)}\,|v|_{H^1(0,\,1)}\,\|v\|_{L_2(0,\,1)}$$

and use it to bound the second term on the right hand side of (2.7).

08 Show the coerivity of $a(w,\,v)$ on the whole space $H^1(0,\,1)$ under the assumptions

$$a_0 > 0\,, \qquad \|b\|_{L_\infty(0,\,1)} < 2\sqrt{a_0\,c_0}\,, \qquad c_0 > 0\,.$$

Hint: Using the estimates above you should be able to obtain

$$a(v,\,v) \;\geq\; q\big(\|v\|_{L_2(0,\,1)},\,|v|_{H^1(0,\,1)}\big)\,,$$

with $q(\xi_0,\,\xi_1) = a_0\,\xi_1^2 - \|b\|_{L_\infty(0,\,1)}\,\xi_1\,\xi_0 + c_0\,\xi_0^2$. Show and use that $q(\xi_0,\,\xi_1) \geq a_0\,C\,\xi_1^2$ and $q(\xi_0,\,\xi_1) \geq c_0\,C\,\xi_0^2$ with $C = 1 - \|b\|^2_{L_\infty(0,\,1)}/(4\,a_0\,c_0)$.

09 Show the coercivity of $a(w,\,v)$ on the space $V_0 = \{v \in H^1(0,\,1) : v(0) = 0$ under the assumptions

$$a_0 > 0\,, \qquad b(x) = b \geq 0\,, \qquad c_0 \geq 0\,,$$

where b is a constant.
Hint: Show and use that

$$b\int_0^1 v'(x)\,v(x)\,dx \;=\; \frac{b}{2}\,v(x)^2\Big|_0^1 \;\geq\; 0 \qquad \forall v \in V_0\,.$$