**01** Show that we can write each linear second order ordinary differential equation

$$-\big(a(x)\,u'(x)\big)' + b(x)\,u'(x) + c(x)\,u(x) = f(x)\,, \tag{1.1}$$

with $a \in C^1(0,\,1)$ and $b,\,c \in C(0,\,1)$, also in the form

$$\bar{a}(x)\,u''(x) + \bar{b}(x)\,u'(x) + c(x)\,u(x) = f(x)\,, \tag{1.2}$$

for suitable functions $\bar{a} \in C^1(0,\,1)$ and $\bar{b} \in C(0,\,1)$. Show also the reverse direction.

**02** Derive the variational formulation for the following two boundary value problems:

$$\text{(a)} \qquad \left\{ \begin{array}{rcll} -u''(x) + u(x) &=& f(x) & \text{for } x \in (0,\,1) \\ u(0) &=& g_0 & \\ u(1) &=& g_1 & \end{array} \right.$$

$$\text{(b)} \qquad \left\{ \begin{array}{rcll} -u''(x) + u(x) &=& f(x) & \text{for } x \in (0,\,1) \\ -u'(0) &=& g_0 - \alpha_0\,u(0) & \\ u(1) &=& g_1 & \end{array} \right.$$

In particular, specify the spaces $V$, $V_0$, and $V_g$, the bilinear form $a(\cdot,\,\cdot)$, and the linear form $\langle F,\,\cdot\rangle$.
*Hint for (b):* Perform integration by parts as usual, substitute $u'(0)$ due to the Robin boundary condition, and collect the bilinear and linear terms accordingly.

**03** Let the sequence $(u_k)_{k\in\mathbb{N}}$ of functions be defined by

$$u_k(x) = \left\{ \begin{array}{ll} 2x & \text{for } x \in \left[0,\,\frac{1}{2} - \frac{1}{2k}\right], \\ 1 - \frac{1}{2k} - 2k\big(x - \frac{1}{2}\big)^2 & \text{for } x \in \left(\frac{1}{2} - \frac{1}{2k},\,\frac{1}{2} + \frac{1}{2k}\right), \\ 2(1-x) & \text{for } x \in \left[\frac{1}{2} + \frac{1}{2k},\,1\right]. \end{array} \right.$$

Show that $u \in C^1[0,\,1]$. Let $u$ be defined by

$$u(x) = \left\{ \begin{array}{ll} 2x & \text{for } x \in \left[0,\,\frac{1}{2}\right], \\ 2(1-x) & \text{for } x \in \left(\frac{1}{2},\,1\right]. \end{array} \right.$$

Find out if $u \in H^1(0,\,1)$ or not and justify your answer. Calculate $\|u_k - u\|_{H^1(0,\,1)}$ (or find a suitable bound for it) and show that

$$\lim_{k\to\infty} \|u_k - u\|_{H^1(0,\,1)} = 0\,.$$

Use these results to show that $(u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $C^1[0,\,1]$ with respect to the $H^1$-norm, but that there exists no limit in $C^1[0,\,1]$.