ÜBUNGEN ZU

NUMERISCHE METHODEN IN DER KONTINUUMSMECHANIK 1

für den 20. 3. 2009

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1. Show for the Green-St.Venant strain tensor $\mathbf{E}(X)$:

$$E_{ij}(X) = \frac{1}{2}\left(\frac{\partial U_j}{\partial X_i}(X) + \frac{\partial U_i}{\partial X_j}(X) + \sum_k \frac{\partial U_k}{\partial X_i}(X)\frac{\partial U_k}{\partial X_j}(X)\right),$$

   where $U(X)$ denotes the displacement in Lagrangian coordinates.

2. Show for the Almansi-Hamel strain tensor $\mathbf{e}(x)$:

$$e_{ij}(x) = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i}(x) + \frac{\partial u_i}{\partial x_j}(x) - \sum_k \frac{\partial u_k}{\partial x_i}(x)\frac{\partial u_k}{\partial x_j}(x)\right),$$

   where $u(x)$ denotes the displacement in Eulerian coordinates.

3. Show the following relation between the Green-St.Venant strain tensor and the Almansi-Hamel strain tensor:

$$\mathbf{E}(X) = \mathbf{F}(X)^T\mathbf{e}(x)\mathbf{F}(X)$$

   with the deformation gradient $\mathbf{F}(X)$, where X and x denote the Lagrangian coordinates and the corresponding Eulerian coordinates, respectively.

4. Consider a three-dimensional steady state flow of an incompressible ideal fluid, whose governing equations are

$$\begin{aligned}(\mathbf{v}\cdot\mathbf{grad})\,\mathbf{v} + \frac{1}{\rho}\,\mathbf{grad}\,p &= \mathbf{f} \quad \text{in } \Omega,\\ \operatorname{div}\mathbf{v} &= 0 \quad \text{in } \Omega.\end{aligned}$$

   Assume that the velocity $\mathbf{v}$ can be represented by a (scalar) potential ϕ:

$$\mathbf{v} = -\,\mathbf{grad}\,\phi. \tag{1}$$

   Show that

$$\mathbf{curl}\,\mathbf{v} = 0$$

   with

$$\mathbf{curl}\,\mathbf{v} = \nabla\times\mathbf{v} = \begin{pmatrix}\dfrac{\partial v_3}{\partial x_2} - \dfrac{\partial v_2}{\partial x_3}\\ \dfrac{\partial v_1}{\partial x_3} - \dfrac{\partial v_3}{\partial x_1}\\ \dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2}\end{pmatrix},$$

   and derive the following relations from the governing equations:

(a) ϕ satisfies the Laplacian equation

$$-\Delta\phi = 0.$$

(b) For a specific force density of the form $\mathbf{f} = -\,\mathbf{grad}\,U$ the so-called Bernoulli equation holds:

$$p + \frac{1}{2}\rho\, v^2 + \rho\, U = \text{constant},$$

where v denotes the Euclidian norm of $\mathbf{v}$.

Hint (this also applies to the following two exercises): All involved functions are assumed to be sufficiently smooth. For sufficiently smooth functions g we have

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} g = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} g.$$

Therefore, more generally,

$$ABg = BAg$$

for all differential operators A, B with constant coefficients, like div, **grad**, curl, **curl**, Δ, as long as the expressions make sense.

For (b) show and use

$$(\mathbf{v}\cdot\mathbf{grad})\,\mathbf{v} = \mathbf{grad}\left(\frac{1}{2}v^2\right) - \mathbf{v}\times\mathbf{curl}\,\mathbf{v}.$$

5. Consider a two-dimensional steady state flow of a Newtonian fluid, whose governing equations are (for simplicity without convective term):

$$\begin{aligned} -\nu\,\Delta\mathbf{v} + \frac{1}{\rho}\,\mathbf{grad}\,p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{v} &= 0 \quad \text{in } \Omega. \end{aligned}$$

Assume that the velocity $\mathbf{v}$ can be represented by a (scalar) stream-function ψ:

$$\mathbf{v} = \mathbf{curl}\,\psi \tag{2}$$

with

$$\mathbf{curl}\,\psi = \begin{pmatrix} \dfrac{\partial\psi}{\partial x_2} \\ -\dfrac{\partial\psi}{\partial x_1} \end{pmatrix}.$$

Show that

$$\operatorname{div}\mathbf{v} = 0,$$

and derive the following relation from the governing equations:

$$\nu\Delta^2\psi = \operatorname{curl}\mathbf{f}$$

with

$$\Delta^2\psi = \Delta(\Delta\psi) \quad \text{and} \quad \operatorname{curl}\mathbf{f} = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}.$$

Hint: Show and use

$$\operatorname{curl}\mathbf{curl}\,\psi = -\Delta\psi.$$

6. Consider a three-dimensional steady state flow of a Newtonian fluid, whose governing equations are (for simplicity without convective term):

$$\begin{aligned} -\nu\,\Delta\mathbf{v} + \frac{1}{\rho}\,\mathbf{grad}\,p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{v} &= 0 \quad \text{in } \Omega. \end{aligned}$$

Assume that the velocity $\mathbf{v}$ can be represented by a (vector-valued) stream-function $\boldsymbol{\Psi}$:

$$\mathbf{v} = \mathbf{curl}\,\boldsymbol{\Psi} \quad \text{with} \quad \operatorname{div}\boldsymbol{\Psi} = 0. \tag{3}$$

Show that

$$\operatorname{div}\mathbf{v} = 0,$$

and derive the following relation from the governing equations:

$$\nu\Delta^2\boldsymbol{\Psi} = \mathbf{curl}\,\mathbf{f}.$$

Hint: Show and use

$$\mathbf{curl}(\mathbf{curl}\,\boldsymbol{\Psi}) = -\Delta\boldsymbol{\Psi} + \mathbf{grad}(\operatorname{div}\boldsymbol{\Psi})$$