# Chapter 1

# Models

## 1.1 Kinematics

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with Lipschitz-continuous boundary $\Gamma = \partial\Omega$. The set $\Omega$ is called the reference configuration and describes, e.g., the initial state or the undeformed state of a continuum (body).

A configuration (or deformation) is a sufficiently smooth, orientation preserving and injective mapping

$$\phi \colon \Omega \longrightarrow \mathbb{R}^3.$$

This mapping describes, e.g., the state of the continuum at some later time or the state of a deformed continuum. The set $\phi(\Omega)$ consists of all points (or particles) $x$ of the form

$$x = \phi(X)$$

with $X \in \Omega$. $X$ are called the material (or Lagrangian) coordinates, $x$ are called the spatial (or Eulerian) coordinates of a particle.

The matrix

$$\mathbf{F}(X) = \nabla\phi(X) = \begin{pmatrix} \frac{\partial \phi_1}{\partial X_1}(X) & \frac{\partial \phi_1}{\partial X_2}(X) & \frac{\partial \phi_1}{\partial X_3}(X) \\ \frac{\partial \phi_2}{\partial X_1}(X) & \frac{\partial \phi_2}{\partial X_2}(X) & \frac{\partial \phi_2}{\partial X_3}(X) \\ \frac{\partial \phi_3}{\partial X_1}(X) & \frac{\partial \phi_3}{\partial X_2}(X) & \frac{\partial \phi_3}{\partial X_3}(X) \end{pmatrix}$$

is called the deformation gradient. Preserving the orientation corresponds to the condition

$$J(X) = \det \nabla\phi(X) > 0 \quad \text{for all } X \in \Omega.$$

The displacement $U \colon \Omega \longrightarrow \mathbb{R}^3$, introduced by

$$U(X) = x - X \quad \text{with} \quad x = \phi(X)$$

measures the deviation from the reference configuration. With

$$x = \phi(X) \quad \text{and} \quad x + \Delta x = \phi(X + \Delta X)$$

we have:

$$\Delta x = \phi(X + \Delta X) - \phi(X) = \nabla\phi(X)\Delta X + o(\Delta X),$$

so

$$\begin{aligned} \|\Delta x\|_{\ell_2}^2 &= \Delta X^T \nabla\phi(X)^T \nabla\phi(X) \Delta X + o(\|\Delta X\|_{\ell_2}^2) \\ &= \Delta X^T \mathbf{C}(x) \Delta X + o(\|\Delta X\|_{\ell_2}^2) \end{aligned}$$

with

$$\mathbf{C}(X) = \mathbf{F}(X)^T \mathbf{F}(X) = \nabla\phi(X)^T \nabla\phi(X).$$

The symmetric tensor $\mathbf{C}(X)$ is called the (right) Cauchy-Green deformation tensor. It describes the local change in distances by the deformation. It can be shown that there is no change in distances, i.e.:

$$\mathbf{C}(X) = I \quad \text{for all } X \in \Omega,$$

if and only if the configuration is a rigid body configuration, i.e.:

$$\phi(X) = QX + a,$$

where $Q$ is an orthogonal matrix with $\det Q = 1$ (describing a rotation) and $a \in \mathbb{R}^3$ (describing a translation).

The deviation of $\mathbf{C}(X)$ from the ideal case $I$ is measured be the symmetric tensor

$$\mathbf{E}(X) = \frac{1}{2}(\mathbf{C}(X) - I),$$

the so called Green-St.Venant strain tensor. Then, of course, we have:

$$\|\Delta x\|_{\ell_2}^2 - \|\Delta X\|_{\ell_2}^2 = 2\,\Delta X^T \mathbf{E}(X) \Delta X + o(\|\Delta X\|_{\ell_2}^2).$$

$\mathbf{E}(X)$ can be expressed directly by the displacement $U(X)$:

$$\mathbf{E}[U](X) = \frac{1}{2}\left(\nabla U(X)^T + \nabla U(X) + \nabla U(X)^T \nabla U(X)\right),$$

or, component-wise:

$$E_{ij}[U](X) = \frac{1}{2}\left(\frac{\partial U_j}{\partial X_i}(X) + \frac{\partial U_i}{\partial X_j}(X) + \sum_k \frac{\partial U_k}{\partial X_i}(X)\frac{\partial U_k}{\partial X_j}(X)\right).$$

Observe the nonlinear relation between $\mathbf{E}$ and $U$.

The displacement can also be introduced in Eulerian coordinates by

$$u(x) = x - X \quad \text{with} \quad x = \phi(X), \text{ i.e. } X = \phi^{-1}(x).$$

Then

$$\Delta X = (\nabla\phi(X))^{-1}\Delta x + o(\Delta x) \quad \text{with } X = \phi^{-1}(x)$$

and, consequently,

$$\|\Delta X\|_{\ell_2}^2 = \Delta x^T \mathbf{c}(x)\Delta x + o(\|\Delta x\|_{\ell_2}^2)$$

with

$$\mathbf{c}(x) = \mathbf{b}(x)^{-1} \quad \text{with} \quad \mathbf{b}(x) = \mathbf{F}(X)\mathbf{F}(X)^T = \nabla\phi(X)\nabla\phi(X)^T.$$

Then

$$\|\Delta x\|_{\ell_2}^2 - \|\Delta X\|_{\ell_2}^2 = 2\,\Delta x^T \mathbf{e}(x)\Delta x + o(\|\Delta x\|_{\ell_2}^2).$$

with

$$\mathbf{e}(x) = \frac{1}{2}(I - \mathbf{c}(x))$$

Finally, it easily follows that

$$\mathbf{e}[u](x) = \frac{1}{2}\left(\nabla u(x)^T + \nabla u(x) - \nabla u(x)^T \nabla u(x)\right).$$

$\mathbf{b}(x)$ is called the Finger deformation tensor or the left Cauchy-Green deformation tensor, $\mathbf{e}(x)$ is called the Almansi-Hamel strain tensor or the Euler strain tensor.

The motion of a continuum (or body) is described by a curve

$$t \mapsto \phi_t.$$

Interpretation: The position $x$ of a point (particle) at time $t$, whose position at time 0 was $X$, is given by

$$x = \phi_t(X) \equiv \phi(X,t).$$

Then the material (or Lagrangian) velocity of this particle as a function of $X$ and $t$ is given by

$$V_t(X) = V(X,t) = \frac{\partial\phi}{\partial t}(X,t),$$

and the material (or Lagrangian) acceleration is given by

$$A_t(X) = A(X,t) = \frac{\partial^2\phi}{\partial t^2}(X,t).$$

Observe the following linear relation between velocity and acceleration:

$$A(X,t) = \frac{\partial V}{\partial t}(X,t).$$

In the Eulerian approach the motion of a particle is described by the spatial velocity (field) $v(x,t)$, where $v(x,t)$ is the velocity of that particle, which passes through $x$ at time $t$, so

$$v_t(x) = v(x,t) = V(X,t) = \frac{\partial \phi}{\partial t}(X,t) \text{ with } x = \phi(X,t).$$

For the spatial acceleration $a(x,t)$ of that particle we obtain:

$$a_t(x) = a(x,t) = A(X,t) = \frac{\partial^2 \phi}{\partial t^2}(X,t) \;\text{ with } x = \phi(X,t).$$

We have for $x = \phi(X,t)$:

$$a(x,t) = \frac{\partial}{\partial t}[v(\phi(X,t),t)] = \frac{\partial v}{\partial t}(x,t) + \sum_i v_i(x,t)\frac{\partial v}{\partial x_i}(x,t).$$

**Notation:** The differential operator $v \cdot \text{grad} = v \cdot \nabla$, given by

$$(v \cdot \text{grad})f = (v \cdot \nabla)f = \sum_{i=1}^{d} v_i \frac{\partial f}{\partial x_i},$$

is called the convective derivative and the differential operator $d/dt$, given by

$$\frac{df}{dt} = \dot{f} = \frac{\partial f}{\partial t} + (v \cdot \text{grad})f,$$

is called the total or material derivative.

With these notations the spatial acceleration can be written in the following form:

$$a(x,t) = \frac{dv}{dt}(x,t) = \frac{\partial v}{\partial t}(x,t) + (v(x,t) \cdot \text{grad})v(x,t) = \frac{\partial v}{\partial t}(x,t) + (v(x,t) \cdot \nabla)v(x,t).$$

Observe that this is a nonlinear relation between velocity and acceleration in the Eulerian approach.

For a given velocity (field) $v(x,t)$ one obtains the trajectories $\phi(X,t)$ of the individual particles as solution of the initial value problem:

$$\begin{aligned} \frac{\partial \phi}{\partial t}(X,t) &= v(\phi(X,t),t), \\ \phi(X,0) &= X. \end{aligned} \tag{1.1}$$

## 1.2 Balance Laws

Let $\omega \subset \Omega$. The set $\omega_t$, given by

$$\omega_t = \{\phi(X,t) \mid X \in \omega\}, \tag{1.2}$$

describes the position of those particles at time $t$, which were in $\omega$ at time $t = 0$.

### 1.2.1 Transport Theorem

Let $F$ be a given function of $x$ and $t$. The Transport Theorem describes the rate change of the quantity

$$\mathcal{F}(t) = \int_{\omega_t} F(x,t)\ dx. \tag{1.3}$$

Namely:

**Theorem 1.1** (Transport-Theorem). *Let $t_0 \in (T_1, T_2)$, let $\omega \subset \Omega$ be a bounded domain with $\overline{\omega}_0 \subset \Omega$, and let $v$ and $F$ be continuously differentiable. Then $\mathcal{F}$ is well-defined and continuously differentiable in an interval $(t_1, t_2) \subset (T_1, T_2)$ with $t_0 \in (t_1, t_2)$ by the equations (1.1), (1.2) and (1.3), and we have:*

$$\frac{d\mathcal{F}}{dt}(t) = \int_{\omega_t} \left[\frac{\partial F}{\partial t}(x,t) + \operatorname{div}(Fv)(x,t)\right]\ dx = \int_{\omega_t} \left[\frac{dF}{dt}(x,t) + F\ \operatorname{div}(v)(x,t)\right]\ dx.$$

**Notation:** The following notation was used in the Transport Theorem: $\operatorname{div} G = \nabla \cdot G$, given by

$$\operatorname{div} G = \nabla \cdot G = \sum_{i=1}^{3} \frac{\partial G_i}{\partial x_i}$$

for a continuously differentiable vector-valued function $G$, is called the divergence of $G$.

**Remark:** With the help of Gauss' Theorem it follows immediately that

$$\frac{d\mathcal{F}}{dt}(t) = \int_{\omega_t} \frac{\partial F}{\partial t}\ dx + \int_{\partial\omega_t} F\ v \cdot n\ ds.$$

Here $n = n(x)$ denotes the outer normal unit vector at a point $x$ on the boundary of $\omega_t$.

### 1.2.2 Conservation of Mass

Let $\rho(x,t)$ denote the mass density of a body at the position $x$ and time $t$. The principle of conservation of mass states that no mass will be generated or destroyed, i. e.:

$$\frac{d}{dt}\int_{\omega_t} \rho(x,t)\ dx = 0.$$

Under appropriate smoothness conditions the Transport Theorem implies:

$$\int_{\omega_t} \left[\frac{\partial \rho}{\partial t}(x,t) + \operatorname{div}(\rho v)(x,t)\right]\ dx = 0$$

for all $t$ and all bounded domains $\omega$ with $\overline{\omega} \subset \Omega$. This results in the following differential equation, the so-called equation of continuity: either in conservative form:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0, \tag{1.4}$$

or, equivalently, in the convective form:

$$\frac{d\rho}{dt} + \rho \operatorname{div} v = 0.$$

In the special case $\rho =$ constant (incompressible fluid) the equation of continuity is given by

$$\operatorname{div} v = 0. \tag{1.5}$$

We have (by the substitution rule)

$$\int_{\omega_t} \rho(x,t)\ dx = \int_{\omega} \rho(\phi(X,t))J(X,t)\ dX.$$

Hence, the conservation of mass in Lagrangian coordinates reads:

$$\frac{d}{dt}\left(\rho(\phi(X,t),t)J(X,t)\right) = 0,$$

Therefore,

$$\rho(x,t) = \frac{1}{J(X,t)}\,\rho_0(X) \quad \text{with } x = \phi(X,t) \quad \text{and } \rho_0(X) = \rho(X,0).$$

### 1.2.3 Balance of Momentum and Angular Momentum

The total (linear) momentum of all particles in $\omega_t$ is given by

$$\int_{\omega_t} \rho(x,t)v(x,t)\ dx.$$

Newton's second law states that the rate of change of the (linear) momentum is equal to the applied forces $F(\omega_t)$, hence

$$\frac{d}{dt}\int_{\omega_t} \rho(x,t)v(x,t)\ dx = F(\omega_t). \tag{1.6}$$

The forces acting on the body can be split into applied body forces $F_V(\omega_t)$ and applied surface forces $F_S(\omega_t)$:

$$F(\omega_t) = F_V(\omega_t) + F_S(\omega_t).$$

If the body forces can be described by a specific force density (force per unit mass) $f(x,t)$, then we obtain the representation

$$F_V(\omega_t) = \int_{\omega_t} \rho(x,t)f(x,t)\ dx.$$

An example of such a force is the force of gravity with $f = (0,0,-g)^T$.

The internal surface forces can be described by a vector $\vec{t}(x,t,n)$ (force per unit area), the so-called Cauchy stress vector:

$$F_S(\omega_t) = \int_{\partial\omega_t} \vec{t}(x,t,n(x))\ ds.$$

Summarizing, we obtain the following balance law for the momentum:

$$\frac{d}{dt}\int_{\omega_t} \rho(x,t)v(x,t)\ dx = \int_{\omega_t} \rho(x,t)f(x,t)\ dx + \int_{\partial\omega_t} \vec{t}(x,t,n(x))\ ds.$$

The total angular momentum of all particles in $\omega_t$ is given by

$$\int_{\omega_t} x \times \rho(x,t)v(x,t)\ dx.$$

Newton's second law states that the rate of change of the angular momentum is equal to the applied torque, so

$$\frac{d}{dt}\int_{\omega_t} x \times \rho(x,t)v(x,t)\ dx = \int_{\omega_t} x \times \rho(x,t)f(x,t)\ dx + \int_{\partial\omega_t} x \times \vec{t}(x,t,n(x))\ ds.$$

These two equations are also called equations of motion, in the steady state case, also the equilibrium conditions.

Under reasonable assumptions it can be shown that the stress vector $\vec{t}(x,t,n) = (t_i(x,t,n))$ can be represented by the so-called Cauchy stress tensor $\sigma = (\sigma_{ij})$ in the following form:

$$t_i(x,t,n) = \sum_j \sigma_{ji}(x,t)\, n_j.$$

Using Gauss' Theorem and the Transport Theorem one obtains for sufficiently smooth functions the following differential equation (in conservative form):

$$\frac{\partial}{\partial t}(\rho v_i) + \operatorname{div}(\rho v_i v) = \sum_j \frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i \tag{1.7}$$

from the balance of momentum, or in convective form

$$\rho\frac{\partial v_i}{\partial t} + \rho v \cdot \operatorname{grad} v_i = \sum_j \frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i \tag{1.8}$$

by using the equation of continuity,

It can be shown that the balance of angular momentum is satisfied if and only if $\sigma$ is symmetric:

$$\sigma^T = \sigma.$$

Therefore, the balance of momentum in convective form can also be written in the following form:

$$\rho\frac{\partial v}{\partial t} + \rho(v\cdot \operatorname{grad})v = \operatorname{div}\sigma + \rho f$$

with

$$\operatorname{div}\sigma = \left(\sum_j \frac{\partial \sigma_{ij}}{\partial x_j}\right)_{i=1,2,3}.$$

So far, the equations of motion have been derived in Eulerian coordinates.

By transforming the integrals one easily obtains the equations of motion in Lagrangian coordinates. We have:

$$\begin{aligned}
\int_{\omega_t} \rho(x,t)v(x,t)\;dx &= \int_{\omega} \rho_0(X)V(X,t)\;dX\\
\int_{\omega_t} \rho(x,t)f(x,t)\;dx &= \int_{\omega} \rho_0(X)F(X,t)\;dX\\
\int_{\partial\omega_t} \sigma(x,t)n(x,t)\;ds &= \int_{\partial\omega} \mathbf{P}(X,t)N(X)\;dS
\end{aligned}$$

with the specific force density $F(X,t)$ in Lagrangian coordinates:

$$F(X,t) = f(x,t) \quad \text{for } x = \phi(X,t),$$

the unit normal vector $N(X)$ in Lagrangian coordinates:

$$\nabla\phi(X,t)^{-T}N(X) = \|\nabla\phi(X,t)^{-T}N(X)\|_{\ell_2}\, n(x,t) \quad \text{for } x = \phi(X,t),$$

and

$$\mathbf{P}(X,t) = J(X,t)\,\sigma(x,t)\nabla\phi(X,t)^{-T} \quad \text{for } x = \phi(X,t),$$

the so-called first Piola Kirchhoff stress tensor.

**Remark:** The last transformation rule is based on Nanson's formula:

$$\int_{\partial\omega_t} \sigma(x,t)n(x,t)\;ds = \int_{\partial\omega} \sigma(x,t)\,J(X,t)\,\nabla\phi(X,t)^{-T}N(X)\;dS.$$

Then one obtains from the balance of momentum the following differential equation in Lagrangian coordinates:

$$\rho_0(X)\frac{\partial^2\phi}{\partial t^2}(X,t) - \operatorname{div}\mathbf{P}(X,t) = \rho_0(X)F(X,t).$$

The balance of angular momentum is satisfied if and only if

$$\mathbf{S}(X,t)^T = \mathbf{S}(X,t)$$

with

$$\mathbf{S}(X,t) = \nabla\phi(X,t)^{-1}\mathbf{P}(X,t) = J(X,t)\,\nabla\phi(X,t)^{-1}\sigma(x,t)\nabla\phi(X,t)^{-T} \quad \text{for } x = \phi(X,t),$$

the so-called second Piola Kirchhoff stress tensor.

The corresponding transformation of the tensors $\mathbf{S} \mapsto \sigma$, given by

$$\sigma(x,t) = \frac{1}{J(X,t)}\nabla\phi(X,t)\mathbf{S}(X,t)\nabla\phi(X,t)^T \quad \text{for } x = \phi(X,t)$$

is called the Piola transformation.

**Remark:** Other balance laws like the balance of energy will not be discussed here.

## 1.3 Constitutive Laws

The equations of motion do not yet completely describe the configuration of a body. Equations for the stress in form of a constitutive laws are necessary.

Two important special cases will be considered here:

### 1.3.1 Elastic Materials

A material is called elastic if there is a constitutive law of the form

$$\mathbf{S}(X) = \hat{\mathbf{S}}(X, \mathbf{E}(X)).$$

For the important sub-class of hyperelastic materials the constitutive law can be represented by an energy functional:

$$\hat{\mathbf{S}}(X, \mathbf{E}) = \frac{\partial \Psi}{\partial \mathbf{E}}(X, \mathbf{E}),$$

where $\Psi(X, \mathbf{E})$ is the so-called stored energy function.

A material is called linearly elastic if

$$\Psi(X, \mathbf{E}) = \frac{1}{2}\sum_{ijkl} C_{ijkl}(X) E_{ij} E_{kl},$$

where the so-called elastic coefficients (or elasticity coefficients) $C_{ijkl}(X)$ (which form the so-called elasticity tensor) have the following properties:

$$C_{ijkl}(X) = C_{klij}(X)$$

and

$$C_{ijkl}(X) = C_{jikl}(X) = C_{jilk}(X).$$

From these conditions it follows that only 21 coefficients can be chosen independently from each other. For the corresponding constitutive law we obtain the linear relations:

$$S_{ij} = \sum_{kl} C_{ijkl}(X) E_{kl}, \tag{1.9}$$

which is called Hooke's law.

An important special case of linearly elastic materials are the St.Venant-Kirchhoff materials (homogenous, isotropic, and linearly elastic materials), for which the constitutive law has the form

$$\mathbf{S} = \lambda \, \mathrm{trace}(\mathbf{E}) \, I + 2\mu \, \mathbf{E}.$$

The parameters $\lambda$ and $\mu$ are called Lamé coefficients. They are related to Young's modulus (or modulus of elasticity) $E$ and Poisson's ratio $\nu$ by

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}$$

and, vice versa

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

It can be shown by arguments from physics that:

$$0 < \nu < \frac{1}{2} \text{ and } E > 0.$$

These conditions are equivalent to

$$\lambda > 0 \text{ and } \mu > 0.$$

For St.Venant-Kirchhoff materials the stored energy function takes the form

$$\Psi(\mathbf{E}) = \frac{\lambda}{2} \, (\mathrm{trace}(\mathbf{E}))^2 + \mu \, \mathrm{trace}(\mathbf{E}^2),$$

so

$$C_{ijkl} = \lambda \, \delta_{ij} \, \delta_{kl} + \mu \, (\delta_{ik} \, \delta_{jl} + \delta_{il} \, \delta_{jk}).$$

### 1.3.2 Newtonian Fluids

Starting point is the following ansatz for the Cauchy stress tensor

$$\sigma = -p \, I + \tau,$$

where $p(x,t)$ denotes the pressure in the fluid at the position $x$ and time $t$ and $\tau$ depends on the first spatial derivative of the velocity field $v(x,t)$.

For a parallel flow (in $x_1$ direction) Newton postulated the linear relation

$$\tau_{21} = \mu \frac{dv_1}{dx_2}$$

for the shear stress $\tau_{21}$. The coefficient $\mu$ is called the dynamic viscosity of the fluid.

Under reasonable assumptions it can be shown that this implies the following form for $\tau$:

$$\tau = \lambda \operatorname{div} v \; I + 2\mu \, \varepsilon(v)$$

with

$$\varepsilon(v) = (\varepsilon(v)_{ij}), \quad \varepsilon(v)_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

Observe that $\operatorname{div} v = \operatorname{trace} \varepsilon(v)$ and the formal similarity to the constitutive law for St. Venant-Kirchhoff materials.

Arguments from physics show that

$$\mu \geq 0 \quad \text{and} \quad \hat{\mu} = \lambda + \frac{2}{3}\mu \geq 0.$$

The coefficient $\hat{\mu}$ is called bulk viscosity. In the following we will assume that $\hat{\mu} = 0$, hence $\lambda = -2\mu/3$. Therefore

$$\sigma = -(p + \frac{2\mu}{3} \operatorname{div} v)\, I + 2\mu \, \varepsilon(v).$$

For $\rho = \text{constant}$, $\mu = \text{constant}$ and with the help of (1.5) ($\operatorname{div} v = 0$) the expressions for the internal surface force can be further simplified:

$$\operatorname{div} \sigma = -\operatorname{grad} p + \mu \, \Delta v,$$

where $\Delta$ denotes the Laplacian operator:

$$\Delta = \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}.$$

## 1.4 Boundary Value and Initial-Boundary Value Problems

For a complete description we need boundary conditions and for time-dependent problems initial conditions.

### 1.4.1 Elastostatics and Elastodynamics

Usually Lagrangian coordinates are used in elasticity.

In typical applications the surface force is prescribed on some part $\Gamma_N$ of the boundary $\Gamma = \partial\Omega$ of $\Omega$, given by its surface force density $T_N(x)$. This results in the boundary condition

$$(\nabla\phi\,\mathbf{S})\,N = T_N \quad \text{for all } x \in \Gamma_N,\ t > 0.$$

For the remaining part $\Gamma_D$ of the boundary we assume that the deformation is known. This leads to the boundary condition

$$\phi = \phi_D \quad \text{for all } X \in \Gamma_D,\ t > 0.$$

As initial conditions usually the initial configuration and the initial velocity are prescribed:

$$\phi = \phi_0, \quad \frac{\partial\phi}{\partial t} = V_0 \quad \text{for } t = 0.$$

Hence we obtain the following initial-boundary value problem of elastodynamics:

$$\begin{aligned}
\rho_0\,\frac{\partial^2\phi}{\partial t^2} - \operatorname{div}(\nabla\phi\,\mathbf{S}) &= \rho_0\,F && \text{in } \Omega,\ t > 0,\\
\mathbf{S} &= \hat{\mathbf{S}}(\mathbf{E}) && \text{in } \Omega,\ t > 0,\\
\mathbf{E} &= \frac{1}{2}(\nabla\phi^T\nabla\phi - I) && \text{in } \Omega,\ t > 0,\\
\phi &= \phi_D && \text{on } \Gamma_D,\ t > 0,\\
(\nabla\phi\,\mathbf{S})\,N &= T_N && \text{on } \Gamma_N,\ t > 0,\\
\phi = \phi_0, \quad \frac{\partial\phi}{\partial t} &= V_0 && \text{in } \Omega,\ t = 0.
\end{aligned}$$

The corresponding time-independent problem leads to the following boundary value problem of elastostatics:

$$\begin{aligned}
-\operatorname{div}(\nabla\phi\,\mathbf{S}) &= \rho_0\,F && \text{in } \Omega,\\
\mathbf{S} &= \hat{\mathbf{S}}(\mathbf{E}) && \text{in } \Omega,\\
\mathbf{E} &= \frac{1}{2}(\nabla\phi^T\nabla\phi - I) && \text{in } \Omega,\\
\phi &= \phi_D && \text{on } \Gamma_D,\\
(\nabla\phi\,\mathbf{S})\,N &= T_N && \text{on } \Gamma_N.
\end{aligned}$$

### 1.4.2 Linear(ized) Elasticity

For small displacements it is justified

- not to distinguish between the Eulerian and the Lagrangian description (in the sequel we will use the Eulerian description), and
- to replace the strain tensor by the linearized strain tensor $\varepsilon$, given by

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right).$$

Then Hooke's law (1.9) can be written in the form

$$\sigma_{ij} = \sum_{kl} C_{ijkl}\,\varepsilon_{kl}$$

or, in short,

$$\sigma = C\,\varepsilon.$$

We obtain the following initial-boundary value problem of linear(ized) elastodynamics:

$$\begin{aligned}
\rho_0 \frac{\partial^2 u}{\partial t^2} - \operatorname{div}\sigma &= \rho_0 f && \text{in } \Omega,\ t > 0,\\
\sigma &= C\,\varepsilon && \text{in } \Omega,\ t > 0,\\
\varepsilon &= \frac{1}{2}(\nabla u^T + \nabla u) && \text{in } \Omega,\ t > 0,\\
u &= u_D && \text{on } \Gamma_D,\ t > 0,\\
\sigma\, n &= t_N && \text{on } \Gamma_N,\ t > 0,\\
u = u_0,\quad \frac{\partial u}{\partial t} &= v_0 && \text{in } \Omega,\ t = 0,
\end{aligned}$$

and the following boundary value problem of linear(ized) elastostatics:

$$\begin{aligned}
-\operatorname{div}\sigma &= \rho_0 f && \text{in } \Omega,\\
\sigma &= C\,\varepsilon && \text{in } \Omega,\\
\varepsilon &= \frac{1}{2}(\nabla u^T + \nabla u) && \text{in } \Omega,\\
u &= u_D && \text{on } \Gamma_D,\\
\sigma\, n &= t_N && \text{on } \Gamma_N.
\end{aligned}$$

For St. Venant-Kirchhoff materials we obtain, in particular,

$$\sigma = \lambda \operatorname{trace}(\varepsilon)\, I + 2\,\mu\,\varepsilon$$

and from constitutive law and the linearized strain-displacement relations it follows that:

$$\begin{aligned} -\operatorname{div}\sigma &= -2\,\mu\,\operatorname{div}\varepsilon(u) - \lambda\,\operatorname{grad}\operatorname{div} u \\ &= -\mu\,\Delta u - (\lambda+\mu)\,\operatorname{grad}\operatorname{div} u. \end{aligned}$$

The corresponding second order differential equations for the displacement $u$ are called Lamé (or Cauchy-Navier) equations.

### 1.4.3 The Navier-Stokes Equations

Usually Eulerian coordinates are used in fluid mechanics. The unknown functions are, e.g., the velocity $v(x,t)$ and the pressure $p(x,t)$.

In typical applications the surface force is prescribed on some part $\Gamma_N$ of the boundary $\Gamma = \partial\Omega$ of $\Omega$, given by its surface force density $t_N(x)$. This results in the boundary condition

$$\sigma n = t_N \quad \text{for all } x \in \Gamma_N,\ t > 0.$$

For the remaining part $\Gamma_D$ of the boundary we assume that the velocity is known. This leads to the boundary condition

$$v = v_D \quad \text{for all } x \in \Gamma_D,\ t > 0.$$

As initial condition usually the initial velocity is prescribed:

$$v = v_0 \quad \text{for } t = 0.$$

For the case $\rho$ = constant and $\mu$ = constant one obtains the equations of motion in conservative form

$$\frac{\partial}{\partial t}(\rho v_i) + \operatorname{div}(\rho v_i v) = -\frac{\partial p}{\partial x_i} + \mu \Delta v_i + \rho f_i, \tag{1.10}$$

or in convective form

$$\rho \frac{\partial v}{\partial t} + \rho\,(v\cdot\operatorname{grad})v = -\operatorname{grad} p + \mu\,\Delta v + \rho f \tag{1.11}$$

or, after dividing by $\rho$:

$$\frac{\partial v}{\partial t} + (v\cdot\operatorname{grad})v = -\frac{1}{\rho}\operatorname{grad} p + \nu \Delta v + f \tag{1.12}$$

with $\nu = \mu/\rho$, the kinematic viscosity. The equations (1.10) or (1.11) or (1.12) are called the Navier-Stokes equations.

In summary, one obtains the following initial-boundary value problem of fluid mechanics:

$$\begin{aligned}
\frac{\partial v}{\partial t} + (v \cdot \operatorname{grad})v - \nu\,\Delta v + \frac{1}{\rho}\operatorname{grad} p = f \quad & \text{in } \Omega,\ t > 0, \\
\operatorname{div} v = 0 \quad & \text{in } \Omega,\ t > 0, \\[1ex]
v = v_D \quad & \text{on } \Gamma_D,\ t > 0, \\
\sigma\ n = t_N \quad & \text{on } \Gamma_N,\ t > 0, \\[1ex]
v = v_0 \quad & \text{in } \Omega,\ t = 0,
\end{aligned}$$

and, for the steady state case, the corresponding boundary value problem:

$$\begin{aligned}
(v \cdot \operatorname{grad})v - \nu\,\Delta u + \frac{1}{\rho}\operatorname{grad} p = f \quad & \text{in } \Omega, \\
\operatorname{div} v = 0 \quad & \text{in } \Omega, \\[1ex]
v = v_D \quad & \text{on } \Gamma_D, \\
\sigma\ n = t_N \quad & \text{on } \Gamma_N.
\end{aligned}$$

**Dimensional analysis:**

Starting from reference values $L^*$, $t^*$, $U^*$ and $p^*$ for the length, the time, the velocity and the pressure new variables are introduced by

$$x_i' = \frac{x_i}{L^*},\ t_i' = \frac{t}{t^*},\ v_i' = \frac{v_i}{U^*},\ p' = \frac{p}{p^*}.$$

By transformation of variables one obtains:

$$\frac{\rho U^*}{t^*}\frac{\partial v_i'}{\partial t'} + \frac{\rho (U^*)^2}{L^*}\sum_{j=1}^{N} v_j' \frac{\partial v_i'}{x_j'} = -\frac{p^*}{L^*}\frac{\partial p'}{\partial x_i'} + \frac{\mu U^*}{(L^*)^2}\Delta v_i' + \rho f,$$

or, after multiplication by $L^*/(\rho(U^*)^2)$

$$\frac{L*}{t^* U^*}\frac{\partial v_i'}{\partial t'} + 1 \cdot \sum_{j=1}^{N} v_j' \frac{\partial v_i'}{x_j'} = -\frac{p^*}{\rho (U^*)^2}\frac{\partial p'}{\partial x_i'} + \frac{\mu}{\rho L^* U^*}\Delta v_i' + f'$$

with $f' = L^*/(U^*)^2 \cdot f$. With the setting $t^* = L^*/U^*$, $p^* = \rho(U^*)^2$ and

$$Re = \frac{\rho L^* U^*}{\mu} = \frac{L^* U^*}{\nu},$$

the so-called Reynolds number, one obtains

$$\frac{\partial v_i}{\partial t} + \sum_{j=1}^{N} v_j \frac{\partial v_i}{x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{Re}\Delta v_i + f. \tag{1.13}$$

For $Re \ll 1$ the viscosity of the flow dominates, for $Re \gg 1$ the flow is dominantly convective. For $Re \to \infty$ one formally obtains the so-called Euler equations:

$$\frac{\partial v}{\partial t} + (v \cdot \operatorname{grad})v + \operatorname{grad} p = f.$$

If the transformed equations are multiplied by $(L^*)^2/(\mu U^*)$, one obtains

$$\frac{\rho(L^*)^2}{\mu t^*}\frac{\partial v_i'}{\partial t'} + \frac{\rho L^* U^*}{\mu}\sum_{j=1}^{N} v_j' \frac{\partial v_i'}{x_j'} = -\frac{p^* L^*}{\mu U^*}\frac{\partial p'}{\partial x_i'} + 1 \cdot \Delta v_i' + f'$$

with $f' = \rho(L^*)^2 f/(\mu U^*)$. With the setting $t^* = (\rho(U^*)^2)/\mu$, $p^* = (\mu U^*)/L^*$ it follows that

$$\frac{\partial v_i}{\partial t} + Re \sum_{j=1}^{N} v_j \frac{\partial v_i}{x_j} = -\frac{\partial p}{\partial x_i} + \Delta v_i + f. \tag{1.14}$$

In this formulation one obtains for $Re = 0$ the so-called Stokes equations:

$$\frac{\partial v}{\partial t} - \Delta v + \operatorname{grad} p = f. \tag{1.15}$$