T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerical Methods for Elliptic Problems"

Tutorial 11 Tuesday, 23 June 2009 (Time : $10^{\underline{15}}$ – $11^{\underline{45}}$ Room : T 911)

2.10 Clément's Interpolator

52 Let $\Omega = (0,\,1)$ and consider the equidistant subdivision into elements $[x_{i-1}, x_i] = [(i-1)h, i\,h]$, $i = 1, \dots, n$. For each node $x_i = i\,h$, $i = 1, \dots, n-1$ we define the local L_2-projection $P_i : L_2(x_{i-1},\, x_{i+1}) \to \mathcal{P}_0(x_{i-1},\, x_{i+1}) = \mathbb{R}$ by

$$\int_{x_{i-1}}^{x_{i+1}} (P_i v)\, q\, dx \;=\; \int_{x_{i-1}}^{x_{i+1}} v\, q\, dx \qquad \forall q \in \mathcal{P}_0(x_{i-1},\, x_{i+1}) \quad \forall v \in L_2(x_{i-1},\, x_{i+1})\,,$$

where $\mathcal{P}_0(x_{i-1},\, x_{i+1})$ are the constant functions on $(x_{i-1},\, x_{i+1})$. Show that

1) $P_i v = \frac{1}{2h}\int_{x_{i-1}}^{x_{i+1}} v(x)\, dx$,

2) $\|v - P_i v\|_{L_2(x_{i-1},x_{i+1})} \le c\,h\,\|v'\|_{L_2(x_{i-1},x_{i+1})} \qquad \forall v \in H^1(x_{i-1},\, x_{i+1})$.

53 Let $V_0 := H_0^1(0,\,1)$ and $V_{0h} := \operatorname{span}\{p^{(j)} : j = 1, \dots, n-1\}$ where $p^{(j)}$ is the nodal basis function associated to the node x_j. We define Clément's interpolator $I_h : L_2(0,\,1) \to V_{0h} \subset V_0$ by

$$(I_h u)(x) := \sum_{j=1}^{n-1} (P_j u)\, p^{(j)}(x) \qquad \text{for } x \in [0,\,1]\,.$$

Show that

$$\|u - I_h u\|_{L_2(0,1)} \;\le\; c\,h\,\|u'\|_{L_2(0,1)} \qquad \forall u \in V_0\,.$$

Hint: Follow your lecture notes. The difference here is that we have boundary conditions! Show (by transformation to the reference element) and use the scaled Friedrichs inequality

$$\|u\|_{L_2(x_0,x_1)} \;\le\; c_F\, h\, |u|_{H^1(x_0,x_1)}\,,$$

with $c_F \neq c_F(h)$.

54 Show that

$$|u - I_h u|_{H^1(0,1)} \;\le\; c\,\|u\|_{H^1(0,1)}\,.$$

Hint: In the construction of the proof follow the previous exercise, and find an estimate for $\|{p^{(k)}}'\|_{L_\infty(0,\,1)}$ in terms of h.

55 Show the estimate

$$\|v - P_j v\|_{L_2(U(x^{(j)}))} \;\leq\; c\, h_j\, |v|_{H^1(U(x^{(j)}))} \qquad \forall v \in H^1(U(x^{(j)}))$$

which is needed in the proof of Lemma 2.18 in the lecture notes.
Hint (there are many ways to prove this; here is a sketch of one possibility): Transform $U(x^{(j)})$ to a domain $\widehat{U}$ of unit size. Then insert $-\overline{\widehat{v}} + \overline{\widehat{v}}$ in the transformed left hand side, with the mean value $\overline{\widehat{v}} = |\widehat{v}|^{-1} \int_{\widehat{U}} \widehat{x}(\xi)\, d\xi$. Finally, use Poincaré's inequality, the Bramble-Hilbert lemma, and transform back.

2.11 A posteriori error estimates

56 Section 2.6.2 in the lecture notes treats the residual error estimator for the Dirichlet problem of Poisson's equation. How do we have to modify this estimator such that it works for the CHIP model problem?