T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerical Methods for Elliptic Problems"

Tutorial 10 Tuesday, 16 June 2009 (Time : $10^{\underline{15}}$ – $11^{\underline{45}}$ Room : T 911)

2.9 Variational Crimes

Consider the one-dimensional BVP to find $u \in V_g = V_0 = H_0^1(0,1)$:

$$\int_0^1 \lambda(x)\, u'(x)\, v'(x) dx \;=\; \int_0^1 f(x)\, v(x)\, dx \qquad \forall v \in V_0\,, \tag{2.48}$$

with

$$f \in L^2(0,\,1)\,, \qquad \lambda \in L^\infty(0,\,1)\,, \qquad \text{and} \qquad \lambda(x) \geq \underline{\lambda} > 0 \quad \forall x \in (0,\,1) \text{ a.e.}$$

Let $\mathcal{F}(\Delta) = \mathcal{P}_1$) and assume an equidistant grid ($x_i = ih$, $i = \overline{0, n+1}, h = 1/(n+1)$, $\delta_r = (x_{r-1}, x_r)$). Now we approximate the bilinear form $a(\cdot,\cdot)$ and the linear form $\langle F, \cdot\rangle$ defined in (2.48) using numerical integration, namely the *midpoint rule*:

$$a_h(u_h,\, v_h) \;=\; \sum_{r=1}^{n+1} h\, \lambda(x_r^\star)\, u_h'(x_r^\star)\, v_h'(x_r^\star) \quad \text{and} \quad \langle F_h,\, v_h\rangle_h \;=\; \sum_{r=1}^{n+1} h\, f(x_r^\star)\, v_h(x_r^\star)\,, \tag{2.49}$$

where $x_r^\star = x_{\delta_r}(\frac{1}{2}) = x_{r-1} + \frac{1}{2}h$. To ensure that these expressions are well-defined we assume (for simplicity) that

$$\lambda,\, f \in W^1_\infty(0,\,1)\,.$$

Let $\widetilde{u}_h$ be such that $a_h(\widetilde{u}_h,\, v_h) = \langle F_h,\, v_h\rangle_h$ for all $v_h \in V_{0h}$. We are interested whether the error $\|u-\widetilde{u}_h\|_{H^1(0,1)}$ obeys the same asymptotics with respect to h than if we compute $a(\cdot,\cdot)$ and $\langle F,\,\cdot\rangle$ exactly. This investigation will be done using Strang's first lemma. Throughout we choose the norm $\|\cdot\|_{V_0} := |\cdot|_{H^1(0,1)}$.

45 Show that the bilinear and linear forms above fulfill the standard assumptions (33) and the additional assumption (34) from the lecture notes; the latter is called *uniform ellipticity* since

$$\exists \mu_3 > 0 :\; a_h(v_h,\, v_h) \;\geq\; \mu_3\, \|v_h\|^2_{V_0}$$

must hold with $\mu_3 \neq \mu_3(h)$!
Hint: For the uniform ellipticity, use that $\lambda \geq \underline{\lambda}$ and that the midpoint rule is exact for some (which?) polynomials.

We can now apply Strang's first lemma which yields

$$\|u - \widetilde{u}_h\|_{V_0} \;\leq\; c\left\{ \inf_{v_h \in V_0} \left[\|u - v_h\|_{V_0} + \sup_{w_h \in V_{0h}} \frac{|a(v_h,\, w_h) - a_h(v_h,\, w_h)|}{\|w_h\|_{V_0}} \right] + \right.$$
$$\left. + \sup_{w_h \in V_{0h}} \frac{|\langle F,\, v_h\rangle - \langle F_h,\, v_h\rangle_h|}{\|w_h\|_{V_0}} \right\}$$

46 For $v_h,\, w_h \in V_{0h}$ prove that

$$|a(v_h,\, w_h) - a_h(v_h,\, w_h)| \;\leq\; c\, h\, |\lambda|_{W^1_\infty(0,\,1)}\, \|v_h\|_{V_0}\, \|w_h\|_{V_0}\,.$$

Hint: Treat each element separately and use that $v_h',\, w_h'$ are constant on each element, so that we are left with $|\int_{\delta_r} \lambda(x)\, dx - h\, \lambda(x_r^*)|$. To get an error bound for this term, use Bramble-Hilbert on the reference element.

47 For $\varphi \in H^1(\delta_r)$ prove that

$$\Big| \int_{\delta_r} \varphi(x)\, dx - h\, \varphi(x_r^*) \Big| \;\leq\; c\, h^{3/2}\, |\varphi|_{H^1(\delta_r)}\,,$$

similarly to the exercises in Tutorial 08. Then set $\varphi = f\, w_h$ and show that

$$|\langle F,\, v_h\rangle - \langle F_h,\, v_h\rangle_h| \;\leq\; c\, h\, \|f\|_{W^{1,\infty}(0,\,1)}\, \|w_h\|_{V_0}\,.$$

48 Show that if $u \in H^2(0,\,1)$ then

$$\|u - \widetilde{u}_h\|_{V_0} \;\leq\; c\, h \left\{ \|u\|_{H^2(0,\,1)} + |\lambda|_{W^1_\infty(0,\,1)} + \|f\|_{W^1_\infty}(0,\,1) \right\}.$$

Hint: In the result of Strang's first lemma, choose $v_h = u_h$ where

$$a(u_h,\, v_h) \;=\; \langle F,\, v_h\rangle \qquad \forall v_h \in V_{0h}\,.$$

Show and use that

$$\|u_h\|_{V_0} \leq c(\lambda)\, \|u\|_{V_0}\,.$$

49 Can we get an estimate similar to **48** if λ and f are only piecewise smooth (e.g., if they have a jump across $x = 1/2$) ?

50** Consider the problem to find $u \in V_0 := H_0^1(\Omega)$ such that

$$\int_\Omega \lambda(x)\, \nabla u(x) \cdot \nabla v(x)\, dx \;=\; \int_\Omega f(x)\, v(x)\, dx \qquad \forall v \in V_0$$

with $\Omega \subset \mathbb{R}^2$. For a discretization we use linear triangular elements and we approximate the integrals with a quadrature rule. Which such rule and which assumptions on the data λ and f are sufficient to conclude an estimate of the form $\|u - \widetilde{u}_h\|_{V_0} \leq c(u,\, f,\, \lambda)\, h$?

The following exercise treats the non-conforming case.

51 Assume the standard assumptions (33), the additional assumptions (39) from the lecture notes, and that $a(\cdot,\, \cdot)$ is symmetric. Suppose $u \in V_g$, $\tilde{u}_h \in V_{gh}$ fulfill

$$a(u,\, v) \;=\; \langle F,\, v\rangle \quad \forall v \in V_0\,, \qquad a_h(\tilde{u}_h,\, v_h) \;=\; \langle F,\, v_h\rangle \quad \forall v_h \in V_{0h}\,.$$

Show that

$$\|u - \tilde{u}_h\|_h \;\leq\; \inf_{v_h \in V_{gh}} \left\{ \|u - v_h\|_h + \sup_{w_h \in V_{0h} \setminus \{0\}} \frac{a_h(u - \tilde{u}_h, w_h)}{\|w_h\|_h} \right\}, \tag{2.50}$$

where $\|\cdot\|_h := \sqrt{a_h(\cdot,\cdot)}$.