T U T O R I A L

“Numerical Methods for the Solution of Elliptic Partial Differential Equations”

to the lecture

“Numerical Methods for Elliptic Problems”

Tutorial 9 Wednesday, 03 June 2009 (Time : $10^{\underline{15}}$ – $11^{\underline{45}}$ Room : T 111)

2.7 Inverse Inequalities

41 Under the assumptions of Lemma 2.11, prove the inverse inequality for a given natural number p, i.e.,

$$\|v_h\|_{L_\infty(\Omega)} \;\le\; c\, h^{-\frac{d}{p}}\, \|v_h\|_{L_p(\Omega)} \qquad \forall v_h \in V_h. \tag{2.42}$$

42 Compute the constant $c_A(\Delta)$ in the inequalty

$$\max_{\xi\in\overline{\Delta}} \Big|\sum_{\alpha\in A} v^{(r,\alpha)} p^{(\alpha)}(\xi)\Big| \;\le\; c_A(\Delta) \Big\|\sum_{\alpha\in A} v^{(r,\alpha)} p^{(\alpha)}(\xi)\Big\|_{L_2(\Delta)}, \tag{2.43}$$

used in the proof of Lemma 2.11, for linear triangular elements ($d = 2$, $k = 1$, $\mathcal{F}(\Delta) = \mathcal{P}_1$).

2.8 L_∞-Error Estimates

43 Consider the V_0-elliptic and the V_0-bounded one-dimensional variational problem

$$\text{Find } u \in V_0 = H_0^1(a,b): \quad a(u,v) \;=\; \langle F, v\rangle \qquad \forall v \in V_0\,, \tag{2.44}$$

with given $F \in V_0^*$. Let us discretize problem (2.44) using linear finite elements on an uniform grid $a = x^{(0)} < ... < x^{(i)} = a + ih < ... < x^{(n)} = b$, $h = (b-a)/n$, i. e. $V_{0h} = \text{span}\{p^{(i)} : i = 1, ..., n-1\}$, $\mathcal{F}(\Delta) = \mathcal{P}_1(\Delta)$, $\Delta = (0,1)$, $\Omega = (a,b)$, $d = 1$. Show that, for $u \in V_0 \cap H^2(a,b)$ and for arbitrary but fixed $y \in [a,b]$, the error estimate

$$|u(y) - u_h(y)| \;\le\; c\, h \inf_{v_h\in V_{0h}} \|G(\cdot,y) - v_h(\cdot)\|_{H^1(a,b)}\, |u|_{H^2(a,b)} \tag{2.45}$$

holds. Here u_h denotes the FE-solution, c is some universial, positive constsant, and $G(x,\, y)$ denotes the Green function of the adjoint problem,

$$G(\cdot,\, y) \in V_0: \quad a(v,\, G(\cdot,\, y)) \;=\; \langle\delta(\cdot - y),\, v\rangle := v(y) \qquad \forall v \in V_0\,. \tag{2.46}$$

44 Prove that, under Assumptions 1 and 2 (for $k = 1$) of Theorem 2.6 (the approximation theorem) and for $u \in W^2_\infty(\Omega)$, the following L_∞-error estimate holds,

$$\inf_{v_h \in V_h} \|u - v_h\|_{L_\infty(\Omega)} \leq c\, h^2 \, \|u\|_{W^2_\infty(\Omega)} \,, \tag{2.47}$$

where c is some universal, positive constant.

Hint: Use Bramble-Hilbert with $p = \infty$ (the statement holds although not proved in the lecture) or use it for $p < \infty$ and perform the limit $p \to \infty$ afterwards.