T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerical Methods for Elliptic Problems"

Tutorial 8 Tuesday, 19 May 2009 (Time : $10^{\underline{15}}$ – $11^{\underline{45}}$ Room : T 911)

2.6 Error Estimates

35 Prove Poincaré's inequality, i.e., there exists a positive constant $\bar{c}_P$ such that

$$\int_\Delta |u|^p dx \;\leq\; \bar{c}_P \Big\{ \Big| \int_\Delta u\, dx \Big|^p + \int_\Delta |\nabla u|^p dx \Big\} \quad \forall u \in W_p^1(\Delta)\,, \tag{2.36}$$

where $p \in \mathbb{N}$ is a given. *Hint:* Apply Sobolev's norm equivalence Theorem 1.3.

36 Let $f \in H^3(\Delta)$ be a given fuction where Δ denotes the reference triangle. Show the existence of a constant $c > 0$ satisfying the estimate

$$\Big| \int_\Delta f(\xi,\eta)\, d\xi\, d\eta - \frac{1}{2} \sum_{i=1}^{3} \alpha_i\, f(\xi_i,\eta_i) \Big| \;\leq\; c\, |f|_{3,\Delta}\,, \tag{2.37}$$

where the weights α_i and the integration points (ξ_i,η_i) $(i = 1,2,3)$ are chosen according to Exercises 26 and 27.

37 Let the bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ be defined by a regular triangulation $\mathcal{T}_h = \{\delta_r : r \in R_h\}$ according to Def. 2.3. Let $f \in H^2(\Omega)$ be a given function. Show that there exists a positive h-independent constant c satisfying the error estimate

$$\Big| \int_\Omega f(x) dx - \sum_{r \in R_h} |\delta_r| f(x_\star^{(r)}) \Big| \;\leq\; c\, h^2 \sum_{r \in R_h} |f|_{2,\delta_r}\,, \tag{2.38}$$

where $|\delta_r| = \text{meas}(\delta_r)$ and $x_\star^{(r)} = x_{\delta_r}(1/3, 1/3)$ denote the area and the center of mass of the triangle δ_r.

38⋆ Using technical tricks one can show that

$$\Big| \int_\Omega f(x) dx - \sum_{r \in R_h} |\delta_r| f(x_\star^{(r)}) \Big| \;\leq\; c\, h^2 |f|_{2,\Omega}\,, \tag{2.39}$$

(which does *not* directly follow from (2.38); a proof is found in [Ciralet]). Show that estimate (2.39) cannot be further improved with respect to the power in h. *Hint:* Choose a simple function f, depending only on *one* of the coordinates, a simple mesh of the unit square, and compute the error.

39 Show that for $d = 1$: $\Omega = (0, 1)$, $k = 1$: $\mathcal{F}(\Delta) = \mathcal{P}_1(\Delta)$, and $u(x) = x^2$ there holds

$$\inf_{v_h \in V_h} \int_0^1 |u'(x) - v_h'(x)|^2 dx = \frac{1}{3} h^2 \,, \tag{2.40}$$

where $V_h = \text{span}\{p^{(i)} : i = 0, 1, ..., n\}$ is defined using continuous affine linear finite elements on the mesh $0 = x^{(0)} < ... < x^{(i)} = ih < ... < x^{(n)} = 1$, $h = 1/n$.

40* Under the assumptions 1 and 2 of the Approximation Theorem 2.6, prove the completeness of the FE-spaces $\{V_h\}_{h \in \Theta}$ in $V = H^1(\Omega)$, i.e.,

$$\lim_{h \to 0} \inf_{v_h \in V_h} ||u - v_h|| = 0 \quad \forall u \in V. \tag{2.41}$$