T U T O R I A L

“Numerical Methods for the Solution of Elliptic Partial Differential Equations”

to the lecture

“Numerical Methods for Elliptic Problems”

Tutorial 4 Thursday, 2 April 2009 (Time : $10^{\underline{15}}$ – $11^{\underline{45}}$ Room : T 911)

1.4 Electromagnetic Fields

17 Show the integral identity

$$\int_\Omega \operatorname{curl}(u) \cdot v \, dx = \int_\Omega u \cdot \operatorname{curl}(v) \, dx + \int_\Gamma u \cdot (v \times n) \, ds$$

for all vector functions u, $v \in [C^1(\bar{\Omega})]^3$, where n denotes the external unit normal on the boundary $\Gamma = \partial\Omega$ of the bounded and sufficiently smooth domain $\Omega \subset \mathbb{R}^3$!

18 Let us consider the following variational problem: Find a vector function $u \in V_g = V_0 := H_0(\operatorname{curl}, \Omega) = H_0(\operatorname{curl})$ satisfying

$$\int_\Omega \left[\frac{1}{\mu} \operatorname{curl}(u) \cdot \operatorname{curl}(v) + \sigma u \cdot v \right] dx = \int_\Omega [J \cdot v + M \cdot \operatorname{curl}(v)] \, dx \quad \forall v \in V_0, \quad (1.14)$$

where $J, M \in [L_2(\Omega)]^3$ are given vector functions and $\mu, \sigma \in L_\infty(\Omega)$ are given uniformly positive and bounded scalar functions, i.e., there exist positive constants $\underline{\mu}, \bar{\mu}, \underline{\sigma}$ and $\bar{\sigma}$, satisfying $\underline{\mu} \le \mu(x) \le \bar{\mu}$ and $\underline{\sigma} \le \sigma(x) \le \bar{\sigma}$ for almost all $x \in \Omega$. Prove that these assumptions already guarantee the existence of a unique solution of the variational problem (1.14).

Which solvability condition must the right hand side fulfill in the case $\sigma = 0$ (magnetostatics) ?

1.5 Mixed Variational Formulations

Hint: Use the slides from `http://www.numa.uni-linz.ac.at/Teaching/LVA/2009s/NuEPDE/` !!!

19* Let us consider the solution $(w, \theta) \in V := H_0^1(\Omega) \times (H_0^1(\Omega))^2$ and $\gamma \in Q := H^{-1}(\operatorname{div}, \Omega)$ of the mixed variational problem

$$a((w,\theta),(v,\phi)) + b((v,\phi),\gamma) = \langle f, (v,\phi) \rangle \quad \forall (v,\phi) \in V, \quad (1.15)$$

$$b((w,\theta),\eta) = \langle g, \eta \rangle \quad \forall \eta \in Q, \quad (1.16)$$

where

$$
\begin{aligned}
a((w,\theta),(v,\phi)) &:= a(\theta,\phi) = \frac{1}{6}\int_\Omega \left[\mu \sum_{i,j=1}^{2} \varepsilon_{ij}(\theta)\,\varepsilon_{ij}(\phi) + \frac{\lambda\mu}{\lambda+2\mu}\mathrm{div}(\theta)\,\mathrm{div}(\phi)\right] dx\,dy,\\
b((w,\theta),\eta) &:= \langle \nabla w - \theta, \eta\rangle_{Q^*\times Q} = (\nabla w - \theta, \eta)_0,\\
\langle f,(v,\phi)\rangle &:= \langle f, v\rangle = (f,v)_0,\\
g &:= 0,\\
H^{-1}(\mathrm{div},\Omega) &:= \left\{\eta \in (H^{-1}(\Omega))^2 : \mathrm{div}(\eta) \in H^{-1}(\Omega)\right\}.
\end{aligned}
$$

Prove that if (w,θ) is a sufficiently smooth solution of problem (1.15)–(1.16), then w satisfies the first biharmonic BVP !

20 Formulate the mixed variational formulation of the mixed BVP

$$
-\Delta u = f \;\text{ in }\; \Omega, \qquad u = 0 \;\text{ on }\; \Gamma_1, \qquad \frac{\partial u}{\partial n} = 0 \;\text{ on }\; \Gamma_2
$$

for Poisson's equation with $f \in L^2(\Omega)$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, and $\Gamma_1 \cup \Gamma_2 = \Gamma$!

21 Let us consider the following mixed formulation: Find $(u,p) \in H_0(\mathrm{curl}) \times H_0^1(\Omega)$ such that

$$
\int_\Omega \nu\,\mathrm{curl}(u)\cdot\mathrm{curl}(v)\,dx + \int_\Omega v\cdot\nabla p\,dx = \int_\Omega J\cdot v\,dx \quad \forall v \in H_0(\mathrm{curl}) \tag{1.17}
$$

$$
\int_\Omega u\cdot\nabla q\,dx = 0 \quad \forall q \in H_0^1(\Omega), \tag{1.18}
$$

where $\nu \in L_\infty(\Omega)$ is a uniformly positive function and $J \in (L_2(\Omega))^3$ is weakly divergence-free, i.e.

$$
\int_\Omega J\cdot\nabla q\,dx = 0 \quad \forall q \in H_0^1(\Omega). \tag{1.19}
$$

Show that p is identical to the zero function ! Therefore u is a weakly divergence-free solution of the magnetostatic problem (1.14) with $\nu = 1/\mu, \sigma = 0, M = 0$.

Hint: $q \in H_0^1(\Omega)$ implies that the tangential derivative $\nabla q \times n$ vanishes on Γ.