T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerics of Elliptic Problems"

Tutorial 2 Tuesday, 17 March 2009 (Time : 10:15 – 11:45, Room : T 911)

1.2 The Linear Elasticity Problem

08 Show that the classical formulation (8) of the linear elasticity problem (for isotropic and homogeneous materials) given in Section 1.2.2 of the lecture is equivalent to Lamé's PDE system

$$-\mu\Delta u(x) - (\lambda+\mu)\nabla \mathrm{div} u(x) = f(x),\ x \in \Omega, \tag{1.6}$$

with the boundary conditions

$$u = \overline{u} \text{ on } \Gamma_1 \quad \text{and} \quad \sigma \cdot n = t \text{ on } \Gamma_2, \tag{1.7}$$

where $f = (f_1, f_2, f_3)^T$, $\overline{u} = (\overline{u}_1, \overline{u}_2, \overline{u}_3)^T$, $t = (t_1, t_2, t_3)^T$ are given vector valued functions, and Δ, ∇, and div denote the vectorial Laplace, the gradient, and the divergence operator, respectively.

09 Show that for the first BVP ($\Gamma_1 = \Gamma$) and for the mixed BVP ($\mathrm{meas}_2(\Gamma_1) > 0$ and $\mathrm{meas}_2(\Gamma_2) > 0$) of the linear elasticity the following properties hold:

1. $a(.,.)$ is symmetric, i.e., $a(u, v) = a(v, u) \quad \forall u, v \in V$,
2. $a(.,.)$ is nonnegative, i.e., $a(v, v) \geq 0 \quad \forall v \in V$,
3. $a(.,.)$ is positive on $V_0 := \{v \in V = [H^1(\Omega)]^3 : v = 0 \text{ on } \Gamma_1\}$, if $\mathrm{meas}_2(\Gamma_1) > 0$, i.e., $a(v, v) > 0 \quad \forall v \in V_0 : v \not\equiv 0$.

The equivalence of VF $(9)_{VF}$ and MP $(9)_{MP}$ then follows from the statements 1. and 2. above according to Section 1.1 of the lecture.

10 Show that, for the first BVP ($\Gamma_1 = \Gamma$) of 3D linear elasticity in the case of an isotropic and homogeneous material, the assumptions of Lax-Milgram-Theorem are fulfilled. Provide constants μ_1 and μ_2 such that

1) $F \in V_0^*$,

2a) $\exists\, \mu_1 = \text{const} > 0 :\ a(v, v) \geq \mu_1 \parallel v \parallel^2_{H^1(\Omega)} \ \ \forall v \in V_0$,

2b) $\exists\, \mu_2 = \text{const} > 0 :\ |a(u, v)| \leq \mu_2 \parallel u \parallel^2_{H^1(\Omega)} \parallel v \parallel^2_{H^1(\Omega)} \ \ \forall u, v \in V_0$.

○ <u>Hints</u> to prove the V_0-ellipticity:

1. $a(v,v) \geq 2\mu \int_\Omega \sum_{i,j=1}^3 (\varepsilon_{ij}(v))^2 dx$,
2. Korn's inequality for the case: $V_0 = [H_0^1(\Omega)]^3$, with $H_0^1(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma\}$,
3. Friedrichs' inequality.

11 Formulate the iterative method (3) from Section 1.1 of the lecture for the first BVP of the linear elasticity in case of 3D homogeneous and isotropic material, i.e.,

$$u_{n+1} = u_n - \rho(JAu_n - JF) \text{ in } V_0 = (H_0^1(\Omega))^3, \tag{1.8}$$

for $n = 0, 1, 2, \ldots$, and given $u_0 \in V_0$. Derive the weak form, i.e., the variational formulation for computing $u_{n+1} \in V_0$. Discuss the following two choices of the norm in V_0,

$$||u||_{V_0}^2 := \int_\Omega |\nabla u|^2 \, dx \quad \forall u \in V_0, \tag{1.9}$$

and

$$||u||_{V_0}^2 := \int_\Omega (|\nabla u|^2 + |u|^2) \, dx \quad \forall u \in V_0. \tag{1.10}$$

12* Let us consider the variational formulation:

$$\text{Find} \quad u \in V_g = V_0 : \quad a(u,v) = \langle F, v \rangle \quad \forall v \in V_0 \tag{1.11}$$

of a plane linear elasticity problem in $\Omega = (0,1) \times (0,1)$, where

$$\begin{aligned} V_0 = \Big\{ \; & u = (u_1, u_2) \in V = [H^1(\Omega)]^2 : \\ & u_1 = 0 \text{ on } \Gamma_2 = \{0\} \times [0,1] \cup \{1\} \times [0,1], \\ & u_2 = 0 \text{ on } \Gamma_1 = [0,1] \times \{0\} \cup [0,1] \times \{1\} \Big\}, \\ a(u,v) &= \int_\Omega D_{ijkl} \varepsilon_{ij}(u) \varepsilon_{kl}(v) \, dx = \int_\Omega \sigma_{kl}(u) \varepsilon_{kl}(v) \, dx, \\ \langle F, v \rangle &= \int_\Omega f_i v_i \, dx + \int_{\Gamma_1} ? \, ds + \int_{\Gamma_2} ? \, ds. \end{aligned}$$

Impose the right natural boundary conditions, and give the classical formulation of (1.11).