T U T O R I A L

"Numerical methods for the solution of elliptic partial differential equations"

to the lecture

"Numerics of elliptic problems"

Tutorial 1 Thursday, 12 March 2009 (Time: 10:15 - 11:45, Room : T 911)

1 Variational formulation of multi-dimensional elliptic BVP

1.1 Scalar elliptic BVP of the second order.

○ In the lectures (Section 1.2.1), we discussed the BVP (classical formulation)

$$\begin{array}{l}
\text{Search } u \in X := C^2(\Omega) \cap C^1(\Omega \cup \Gamma_2 \cup \Gamma_3) \cap C(\Omega \cup \Gamma_1) : \\
-\sum\limits_{i,j=1}^{m} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) + \sum\limits_{i=1}^{m} a_i(x)\frac{\partial u}{\partial x_i} + a(x)u(x) = f(x), x \in \Omega \\
+\text{BC:} \quad \bullet \quad u(x) = g_1(x), x \in \Gamma_1 \\
\qquad\quad \bullet \quad \frac{\partial u}{\partial N} := \sum\limits_{i,j=1}^{m} a_{ij}(x)\frac{\partial u(x)}{\partial x_j} n_i(x) = g_2(x), x \in \Gamma_2 \\
\qquad\quad \bullet \quad \frac{\partial u}{\partial N} + \alpha(x)u(x) = \underbrace{g_3(x)}_{\alpha(x)u_A(x)}, x \in \Gamma_3
\end{array} \tag{1.1}$$

and derived its variational formulation

$$\begin{array}{lcl}
\multicolumn{3}{l}{\text{Search } u \in V_g : a(u,v) = < F, v > \quad \forall v \in V_0} \\
\multicolumn{3}{l}{\text{with}} \\
a(u,v) & := & \int\limits_{\Omega}(\sum\limits_{i,j=1}^{m} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + \sum\limits_{i=1}^{m} a_i \frac{\partial u}{\partial x_i} v + auv)\,dx + \int\limits_{\Gamma_3} \alpha uv\,ds, \\
< F, v > & := & \int\limits_{\Omega} fv\,dx + \int\limits_{\Gamma_2} g_2 v\,ds + \int\limits_{\Gamma_3} g_3 v\,ds \\
\multicolumn{3}{l}{V_g := \{v \in V = W_2^1(\Omega) : v = g_1 \text{ on } \Gamma_1\}} \\
\multicolumn{3}{l}{V_0 := \{v \in V : v = 0 \text{ on } \Gamma_1\}}
\end{array} \tag{1.2}$$

under the assumptions

$$\left.\boxed{\begin{array}{ll}1) & a_{ij}, a_i, a \in L_\infty(\Omega), \alpha \in L_\infty(\Gamma_3) \\ 2) & f \in L_2(\Omega), g_i \in L_2(\Gamma_i), i = 2,3 \\ 3) & g_1 \in H^{\frac{1}{2}}(\Gamma_1), \text{ i.e., } \exists \tilde{g}_1 \in H^1(\Omega) : \tilde{g}_1|_{\Gamma_1} = g_1. \\ 4) & \Omega \subset \mathbf{R}^m (\text{bounded}) : \Gamma = \partial\Omega \in C^{0,1} \text{ (Lipschitz domain)} \\ 5) & \text{uniform ellipticity:} \\ & \left.\begin{array}{l}\sum\limits_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j \geq \bar{\mu}_1|\xi|^2 \quad \forall \xi \in \mathbf{R}^m \\ a_{ij}(x) = a_{ji}(x) \quad \forall i,j = \overline{1,m}\end{array}\right\} \forall \text{ a.e. } x \in \Omega.\end{array}}\right\} \qquad (1.3)$$

01 Formulate the classical assumptions which we have to impose on the data $\{ a_{ij}, a_i, a, \alpha, f, g_i, \Omega \text{ resp. } \partial\Omega \}$ for (1.1) !

02 Provide sufficient conditions in order to ensure that a generalized solution $u \in V_g \cap X \cap W_2^2(\Omega)$ of (**) is also a solution of (*) in the classical sense !

$$(*) \quad \left\{ \begin{array}{l} \text{Search } u \in X = C^2(\Omega) \cap C(\bar{\Omega}) : \\ -\Delta u(x) = f(x), x \in \Omega \subset \mathbf{R}^m \text{ (bounded)}, \\ u(x) = g(x), x \in \Gamma = \partial\Omega \end{array} \right.$$

?⇓⇑?

$$(**) \quad \left\{ \begin{array}{l} \text{Search } u \in V_g = \{v \in V = H^1(\Omega) : v = g \text{ on } \Gamma\} : \\ \underbrace{\int\limits_\Omega \nabla^T u \nabla v \, dx}_{=a(u,v)} = \underbrace{\int\limits_\Omega f v \, dx}_{=<F,v>} \quad \forall v \in V_0 = \overset{o}{H}{}^1(\Omega) = \overset{o}{W}{}_2^1(\Omega) \end{array} \right.$$

03 Provide sufficient conditions in order to ensure that a generalized solution $u \in V_g \cap X \cap W_2^2(\Omega)$ of (1.2) is also a solution of (1.1) in the classical sense !

04 Show that in the following cases a) and b) the assumption of the Lax-Milgram-Theorem are satisfied, and derive μ_1 and μ_2 !

a) In addition to (1.3) the following assumptions hold: $a_i = 0$, $a(x) \geq 0$ for almost all $x \in \Omega$, $\alpha(x) \geq 0$ for almost all $x \in \Gamma_3$, $\text{meas}_{m-1}(\Gamma_1) > 0$.

b) In addition to (1.3) the following assumptions hold: $a_i = 0$, $a = 0$, $\alpha(x) \geq \underline{\alpha} = \text{const} > 0$ for almost all $x \in \Gamma_3$, $\text{meas}_{m-1}(\Gamma_3) > 0$, $\Gamma_1 = \emptyset$.

05 In addition to the assumption (1.3) let us assume that $a(x) \geq \underline{a} = \text{const} > 0$ for almost all $x \in \Omega$, $\Gamma_1 = \Gamma_3 = \emptyset$, and $a_i \not\equiv 0$. Provide sufficient conditions for the coefficients a_i such that the assumptions of the Lax-Milgram-Theorem are satisfied.

○ Hint:

For estimating the convection term $\sum_{i=1}^{m} \int_{\Omega} a_i \frac{\partial u}{\partial x_i} v \, dx$, make use of the ε-inequality

$$|ab| \leq \frac{1}{2\varepsilon} a^2 + \frac{\varepsilon}{2} b^2, \quad \forall a, b \in \mathbf{R}^1 \quad \forall \varepsilon > 0 \, !$$

06* Derive the variational formulation for the pure Neumann-problem for the Poisson equation

$$-\Delta u(x) = f(x), \ \forall x \in \Omega \ \text{ and } \ \frac{\partial u}{\partial n}(x) = 0, \ \forall x \in \Gamma = \partial\Omega, \tag{1.4}$$

and discuss the question of the existence and the uniqueness of the generalized solution of the pure Neumann-problem (1.4) !

○ Hint:

Obviously, $u(x) + c$ with arbitrary constant $c \in \mathbf{R}^1$ solves (1.4) provided that u is the solution of the BVP (1.4). There are the following ways to analyze the existence properties:

1) Set up the variational formulation in $V = H^1(\Omega)$ and apply Fredholm's Theory !
2) Set up the variational formulation in the factor-space $V = H^1(\Omega)|_{\text{ker}}$ with ker= $\{c : c \in \mathbf{R}^1\} = \mathbf{R}^1$ and apply the Lax-Milgram-Theorem !

07* Derive the variational formulation of the Dirichlet problem for the Helmholtz equation

$$-\Delta u(x) - \omega^2 u(x) = f(x), \ \forall x \in \Omega \ \text{ and } \ u(x) = 0, \ \forall x \in \Gamma = \partial\Omega, \tag{1.5}$$

Then discuss the problem of the existence and the uniqueness of the generalized solution of the BVP (1.5), where ω^2 is a given positive constant.