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Domain Decomposition Methods

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1 Introduction

These notes aim at the construction of preconditioners B for solving systems of grid equations approximating elliptic boundary value problems in domains with complex geometry. A preconditioner B can be used, for example, in iterative processes of the following form:

$$B(u^{k+1} - u^k) = -\tau^k (Au^k - f), \tag{1.1}$$

where A is the stiffness matrix of the original system of grid equations. The convergence rate of the iterative process (1.1) depends on the constants c_1 and c_2 in the spectral equivalence inequalities

$$c_1(Bu, u) \leq (Au, u) \leq c_2(Bu, u), \tag{1.2}$$

which should be valid for any vector u. Here, we assume that A and B are symmetric positive definite matrices. In [8], a technique has been suggested for constructing the preconditioner B in the case of uniform grids of a rectangle. In addition, the constants c_1 and c_2 from (1.2) are independent of the mesh size, and, in order to perform the multiplication of B^{-1} with some vector, it is necessary to solve the system of grid equations corresponding to the five-point approximation of the Laplace operator on a uniform grid of a rectangle. The construction of a preconditioner B with similar characteristics in the case of boundary value problems in domains with complex geometry is of great interest.

The most efficient preconditioners for solving boundary value problems in domains with complex geometry can be constructed, as a rule, by 'simplifying' the geometry of the original domain. Here, we can point out two approaches. The first approach is to partition the original domain into simpler subdomains (domain decomposition methods), and the second approach is to embed the original domain in a domain of some canonical form, for example, a rectangle in the two-dimensional case and a parallelepiped in the three-dimensional case, by introducing additional equations (the fictitious domain method and its matrix counterparts) [9, 13, 16, 15, 17, 32, 36, 38].

Of the first group of methods, the so-called Additive Schwarz Method (ASM) is very effective. The classical overlapping domain decomposition method was first proposed by H. A. Schwarz in [40]. As a solution method for finite element equations, the ASM was suggested in [21]. Domain decomposition methods are the subject of the textbooks [34, 35, 41, 44]. In these notes, results from [21, 22, 23, 25, 26, 27, 29] are used.

In the second group of methods, major gains have been obtained for problems with natural boundary conditions [1]. Using the matrix counterpart of the fictitious domain method, we have managed to construct a preconditioner B such that the constants c_1 and c_2 from (1.2) are independent of h, and the main operation in performing multiplication of B^{-1} with a vector consists in solving the five-point finite difference counterpart of the Laplace operator in the rectangle. Later, in [18, 19], the so-called non-symmetric augmentation of the original system of grid equations was proposed to solve the Dirichlet problem, as well as an iterative process for solving this augmented system of equations whose convergence rate is independent of h. Moreover, the main operation in one iteration step of the iterative process is the solution of the five-point finite difference counterpart of the Laplace operator in the rectangle. Some preconditioner using this idea (not optimal), has been constructed before. Finally, in [2, 14, 20, 24], the case of mixed boundary value problems was considered for second-order elliptic equations. The authors suggested an iterative process whose convergence rate is logarithmically dependent on h. It was the Dirichlet boundary condition that made it impossible to avoid the dependence on h. The most flexible approach to the construction of fictitious domain type preconditioning operators in domains with complex geometries is provided by the Fictitious Space Lemma, which was presented in [28, 30]. This lemma gives the possibility to use "convenient" fictitious (auxiliary) spaces equipped with "convenient" norms. In particular, instead of a direct solver for the five-point approximation of the Laplace operator, BPX-like multilevel preconditioners can be used.

2 Domain Decomposition

Let Ω be an L-shaped domain with the boundary $\Gamma = \partial\Omega$, and let Ω be decomposed into two rectangles Ω_1 and Ω_2 with the common boundary γ. We now consider the Dirichlet problem for the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \Gamma$$

as model problem. Let $H_0^1(\Omega) = \{u \in H^1(\Omega) \,|\, u = 0 \text{ on } \Gamma\}$ the subspace of all functions from the Sobolev space $H^1(\Omega)$ vanishing on the boundary Γ. Then the weak formulation reads as follows: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega (\nabla u, \nabla v) d\Omega = \int_\Omega f v \, d\Omega \; \forall v \in H_0^1(\Omega).$$

We assume that Ω^h is an uniform triangulation of Ω and define the discrete space $H_h(\Omega) = \{u^h \in H_0^1(\Omega) \,|\, u^h = \text{piecewise-linear}\}$. The finite element function $u^h \in H_h(\Omega)$ can be identified with the vector $\bar{u} = [u_1, \ldots, u_N]^\top$, where $u_i = u^h(z_i)$ are the values at the nodes z_i. The finite element approximation to the weak formulation results in the finite element equations of the form

$$A\bar{u} = \bar{f}, \tag{2.1}$$

where the stiffness matrix A and the load vector $\bar{f}$ are defined by the identities

$$(A\bar{u}, \bar{v}) = \int_\Omega (\nabla u^h, \nabla v^h) d\Omega \quad \text{and} \quad (\bar{f}, \bar{v}) = \int_\Omega f v^h d\Omega,$$

respectively.

The vector $\bar{u}$ can be decomposed into three groups, that is, $\bar{u} = [\bar{u}_0, \bar{u}_1, \bar{u}_2]^\top = \left[\bar{u}_0^\top, \bar{u}_1^\top, \bar{u}_2^\top\right]^\top$, where $\bar{u}_0$, $\bar{u}_1$, and $\bar{u}_2$ are corresponding to γ, Ω_1, and Ω_2, respectively. Apparently, we get

$$A\bar{u} = \begin{bmatrix} A_0 & A_{01} & A_{02} \\ A_{10} & A_1 & 0 \\ A_{20} & 0 & A_2 \end{bmatrix} \begin{bmatrix} \bar{u}_0 \\ \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = \begin{bmatrix} \bar{f}_0 \\ \bar{f}_1 \\ \bar{f}_2 \end{bmatrix} = \bar{f}.$$

We observe that A_i corresponds to the Dirichlet problem in Ω_i, that is, $A_i \longleftrightarrow -\Delta_{\Omega_i}$, for $i = 1, 2$. From the second and the third equation, we obtain

$$\bar{u}_1 = A_1^{-1}\bar{f}_1 - A_1^{-1}A_{10}\bar{u}_0, \tag{2.2}$$

$$\bar{u}_2 = A_2^{-1}\bar{f}_2 - A_2^{-1}A_{20}\bar{u}_0. \tag{2.3}$$

Substituting (2.2) and (2.3) into the first equation, we get

$$(A_0 - A_{01}A_1^{-1}A_{10} - A_{02}A_2^{-1}A_{20})\bar{u}_0 = \bar{f}_0 - A_{01}A_1^{-1}\bar{f}_1 - A_{02}A_2^{-1}\bar{f}_2. \qquad (2.4)$$

Introducing the Schur complement matrix

Note 1: We wrote Schur complement everywhere (instead of Schur-complement and Schur-Complement)

$$S = A_0 - A_{01}A_1^{-1}A_{10} - A_{02}A_2^{-1}A_{20}$$

and the vectors

$$\phi = \bar{u}_0 \quad \text{and} \quad \psi = \bar{f}_0 - A_{01}A_1^{-1}\bar{f}_1 - A_{02}A_2^{-1}\bar{f}_2,$$

we can rewrite system (2.4) in the abbreviated form

$$S\phi = \psi. \qquad (2.5)$$

If we can find a good preconditioner Σ for S, then the solution ϕ can efficiently be approximated by an iterative method, e.g. by the Richardson iteration

$$\Sigma(\phi^{k+1} - \phi^k) = -\tau_k(S\phi^k - \psi),$$

where τ_k denotes some suitably chosen relaxation parameter. The correspondence between an approximate solution of the Schur complement problem (2.5) with the original problem (2.1) is given by the following lemma.

Lemma 2.1 *If $\|\phi^n - \phi\|_S = \epsilon$, then $\|\bar{u}^n - \bar{u}\|_A = \epsilon$, where the components of the vector $\bar{u}^n$ are given by $\bar{u}_0^n = \varphi^n$, $\bar{u}_1^n = A_1^{-1}(\bar{f}_1 - A_{10}\phi^n)$ and $\bar{u}_2^n = A_2^{-1}(\bar{f}_2 - A_{20}\phi^n)$.*

Proof. Using (2.2) and (2.3), we obtain the equations

$$\begin{aligned}
\|\bar{u}^n - \bar{u}\|_A^2 &= (A(\bar{u}^n - \bar{u}), \bar{u}^n - \bar{u}) \\
&= \left(\begin{bmatrix} A_0 & A_{01} & A_{02} \\ A_{10} & A_1 & 0 \\ A_{20} & 0 & A_2 \end{bmatrix} \begin{bmatrix} \bar{u}_0^n - \bar{u}_0 \\ \bar{u}_1^n - \bar{u}_1 \\ \bar{u}_1^n - \bar{u}_2 \end{bmatrix}, \begin{bmatrix} \bar{u}_0^n - \bar{u}_0 \\ \bar{u}_1^n - \bar{u}_1 \\ \bar{u}_1^n - \bar{u}_2 \end{bmatrix} \right) \\
&= \left(\begin{bmatrix} A_0\bar{u}_0^n + A_{01}\bar{u}_1^n + A_{02}\bar{u}_2^n - \bar{f}_0 \\ A_{10}\bar{u}_0^n + A_1\bar{u}_1^n - \bar{f}_1 \\ A_{20}\bar{u}_0^n + A_2\bar{u}_2^n - \bar{f}_2 \end{bmatrix}, \begin{bmatrix} \bar{u}_0^n - \bar{u}_0 \\ \bar{u}_1^n - \bar{u}_1 \\ \bar{u}_1^n - \bar{u}_2 \end{bmatrix} \right) \\
&= \left(\begin{bmatrix} S\bar{u}_0^n - \psi \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \bar{u}_0^n - \bar{u}_0 \\ \bar{u}_1^n - \bar{u}_1 \\ \bar{u}_1^n - \bar{u}_2 \end{bmatrix} \right) = \left(\begin{bmatrix} S\bar{u}_0^n - \psi \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \varphi^n - \varphi \\ 0 \\ 0 \end{bmatrix} \right) = \epsilon,
\end{aligned}$$

which proves the lemma. □

There are some interesting facts about S. The first is that though, S is an interface problem, it is closely related to the entire problem. The second is that the quadratic form $(S\phi, \phi)$ is equivalent to some trace norm. In order to obtain this, we consider

$$\begin{aligned}
(A\bar{u}, \bar{u}) &= \int_{\Omega} |\nabla u^h|^2 \, d\Omega \\
&= \int_{\Omega_1} |\nabla u^h|^2 \, d\Omega_1 + \int_{\Omega_2} |\nabla u^h|^2 \, d\Omega_2 \\
&= \left(A^{(1)} \begin{bmatrix} \bar{u}_0 \\ \bar{u}_1 \end{bmatrix}, \begin{bmatrix} \bar{u}_0 \\ \bar{u}_1 \end{bmatrix} \right) + \left(A^{(2)} \begin{bmatrix} \bar{u}_0 \\ \bar{u}_2 \end{bmatrix}, \begin{bmatrix} \bar{u}_0 \\ \bar{u}_2 \end{bmatrix} \right),
\end{aligned}$$

where $A^{(i)} = \begin{bmatrix} A_0^{(i)} & A_{0i} \\ A_{i0} & A_i \end{bmatrix}$ is just the discrete Laplacian $-\Delta_{\Omega_i}$, which satisfies the Dirichlet condition on $\partial\Omega_i \setminus \gamma$ and the Neumann condition on γ, $i = 1, 2$. Let $S_i = A_0^{(i)} - A_{i0} A_i^{-1} A_{i0}$. Then we have $S = S_1 + S_2$ and $A_0 = A_0^{(1)} + A_0^{(2)}$. Moreover,

$$\begin{aligned}
\inf_{u_1} \left(A^{(1)} \begin{bmatrix} \phi \\ u_1 \end{bmatrix}, \begin{bmatrix} \phi \\ u_1 \end{bmatrix} \right) &= \inf_{u_1} ((A_0^{(1)}\phi, \phi) + (A_1 u_1, u_1) + 2(A_{10}\phi, u_1)) \\
&= (A_0^{(1)}\phi, \phi) + \inf_{u_1} ((A_1 u_1, u_1) - 2(-A_{10}\phi, u_1)).
\end{aligned}$$

The quadratic form $(A_1 u_1, u_1) - 2(-A_{10}\phi, u_1)$ attains its minimum at $A_1 u_1 = -A_{10}\phi$, that is, $u_1 = -A_1^{-1} A_{10}\phi$. Thus,

$$\begin{aligned}
\inf_{u_1} \left(A^{(1)} \begin{bmatrix} \phi \\ u_1 \end{bmatrix}, \begin{bmatrix} \phi \\ u_1 \end{bmatrix} \right) &= (A_0^{(1)}\phi, \phi) + (A_{10}\phi, A_1^{-1} A_{10}\phi) - 2(A_{10}\phi, A_1^{-1} A_{10}\phi) \\
&= (A_0^{(1)}\phi, \phi) - (A_{01} A_1^{-1} A_{10}\phi, \phi) \\
&= (S_1\phi, \phi).
\end{aligned}$$

Hence we obtain the following relation:

$$(S\phi, \phi) = \inf_{u_h \in H_h(\Omega_1), u_h|_\gamma = \phi_h} |u^h|^2_{H^1(\Omega_1)} + \inf_{u_h \in H_h(\Omega_2), u_h|_\gamma = \phi_h} |u^h|^2_{H^1(\Omega_2)}.$$

In fact, the infimum occurs at u^h which solves the discrete Laplacian problem:

$$\begin{aligned}
-\Delta_h u^h &= 0, \quad \text{in } \Omega_i, \\
u^h &= 0, \quad \text{on } \partial\Omega_i \setminus \gamma, \\
u^h &= \phi^h, \quad \text{on } \gamma.
\end{aligned}$$

Let $\Gamma^h = \partial\Omega^h$, $\{z_i\} = \{x_i\}$ be the set of nodes contained in Γ^h, $I_i = [z_i, z_{i+1}]$, and $h_i = |z_{i+1} - z_i|$. Since Ω^h is shape regular, there exist $c_i \neq c_i(h)$ such that $c_1 \leq h_i/h_{i+1} \leq c_2$. Now, we define the discrete norms corresponding to the continuous trace norms by

$$\|\phi^h\|^2_{L_{2,h}(\Gamma^h)} := \sum_{z_i \in \Gamma^h} \phi^h(z_i)^2 h_i$$

and

$$|\phi^h|^2_{H_h^{1/2}(\Gamma^h)} := \sum_{z_i \in \Gamma^h} \sum_{z_j \in \Gamma^h, i \neq j} \frac{(\phi^h(z_i) - \phi^h(z_j))^2}{|z_i - z_j|^2} h_i h_j.$$

Lemma 2.2 *The above discrete norms are equivalent to $\|\cdot\|_{L_2(\Gamma^h)}$ and $|\cdot|_{H^{1/2}(\Gamma^h)}$, respectively, where the equivalence constants are independent of h.*

Proof. For the sake of simplicity we assume that Γ^h is the unit interval $[0,1]$ with the nodes $\{x_i\}$. For the $L_2(\Gamma^h)$ case, it is easy to prove the equivalence by observing

$$\|\phi^h\|_{L_2(\Gamma)} = \sum_{I_i} \|\phi^h\|^2_{L_2(I_i)}.$$

Let us consider the $H^{1/2}(\Gamma_h)$ case. Then, we have

$$\int_\Gamma \int_\Gamma \frac{(\phi^h(x) - \phi^h(y))^2}{|x-y|^2} dxdy = \sum_i \sum_j \int_{I_i} \int_{I_j} \frac{(\phi^h(x) - \phi^h(y))^2}{|x-y|^2} dxdy.$$

The above equation can be treated in the following three different cases:

- Case 1) Let $i = j$. On the interval $I_i = [x_i, x_{i+1}]$, ϕ^h is a linear function, in fact,

$$\phi^h(x) = \phi_i + \frac{\phi_{i+1} - \phi_i}{h_i} x.$$

Therefore,

$$\int_{I_i} \int_{I_j} \frac{(\phi^h(x) - \phi^h(y))^2}{|x-y|^2} dxdy$$
$$= \int_{x_i}^{x_{i+1}} \int_{x_i}^{x_{i+1}} \frac{(\phi_i + \frac{\phi_{i+1}-\phi_i}{h_i} x - \phi_i - \frac{\phi_{i+1}-\phi_i}{h_i} y)^2}{|x-y|^2} dxdy = (\phi_{i+1} - \phi_i)^2.$$

- Case 2) Let $i + 1 < j$. Then we have, $\forall x \in I_i, \forall y \in I_j$,

$$|x_{i+1} - x_j| \leq |x - y| \leq |x_{j+1} - x_j|,$$

and, therefore,

$$\int_{I_i} \int_{I_j} \frac{(\phi^h(x) - \phi^h(y))^2}{|x-y|^2} dxdy \simeq \frac{1}{|x_{i+1} - x_j|^2} \int_{x_j}^{x_{j+1}} \int_{x_i}^{x_{i+1}} (\phi^h(x) - \phi^h(y))^2 \, dxdy.$$

Let $x = x_i + h_i x'$ and $y = x_j + h_j y'$. Then the above equation is equal to

$$\frac{1}{|x_{i+1} - x_j|^2} h_i h_j \int_0^1 \int_0^1 (\tilde{\phi}^h(x') - \tilde{\phi}^h(y'))^2 dx'dy'.$$

We now define bilinear forms A and B by,

$$
\begin{aligned}
&(A[\phi_i, \phi_{i+1}, \phi_j, \phi_{j+1}]^\top, [\phi_i, \phi_{i+1}, \phi_j, \phi_{j+1}]^\top) \\
&\quad = \int_0^1 \int_0^1 (\tilde{\phi}^h(x') - \tilde{\phi}^h(y'))^2 dx'dy', \\
&(B[\phi_i, \phi_{i+1}, \phi_j, \phi_{j+1}]^\top, [\phi_i, \phi_{i+1}, \phi_j, \phi_{j+1}]^\top) \\
&\quad = (\phi_i - \phi_j)^2 + (\phi_i - \phi_{i+1})^2 + (\phi_{i+1} - \phi_j)^2 + (\phi_j - \phi_{j+1})^2.
\end{aligned}
$$

By an easy calculation, we get

$$
\begin{aligned}
&(A[\phi_i, \phi_{i+1}, \phi_j, \phi_{j+1}]^\top, [\phi_i, \phi_{i+1}, \phi_j, \phi_{j+1}]^\top) \\
&= \frac{2}{3}(\phi_i^2 + \phi_i\phi_{i+1} + \phi_{i+1}^2) + \frac{2}{3}(\phi_j^2 + \phi_i\phi_{j+1} + \phi_{j+1}^2) - 2(\phi_i + \phi_{i+1})(\phi_j + \phi_{j+1}).
\end{aligned}
$$

We observe that Ker$A$ = Ker$B$ = $\{\phi \,|\, \phi_i = \phi_{i+1} = \phi_j = \phi_{j+1}\}$. Therefore, there exist constants $c_1, c_2 \neq c(h)$ such that

$$
c_1(A\phi, \phi) \leq (B\phi, \phi) \leq c_2(A\phi, \phi).
$$

- Case 3) Let $i + 1 = j$. Let $x = x_{i+1} - h_i x'$ and $y = x_{i+1} + h_{i+1} y'$. Then we have

$$
\int_{I_i} \int_{I_j} \frac{(\phi^h(x) - \phi^h(y))^2}{|x - y|^2} dxdy = h_i h_{i+1} \int_0^1 \int_0^1 \frac{(\tilde{\phi}^h(-x') - \tilde{\phi}^h(y'))^2}{|h_i x' + h_{i+1} y'|^2} dx'dy'.
$$

Assume that $h_{i+1} \geq h_i$. Then $\frac{h_{i+1}}{h_i} = a \geq 1$, so that $x' + y' \leq x' + ay' \leq a(x' + y')$. Therefore, the above equation is equal to

$$
\begin{aligned}
&\frac{h_{i+1}}{h_i} \int_0^1 \int_0^1 \frac{(\tilde{\phi}^h(-x') - \tilde{\phi}^h(y'))^2}{|x' + ay'|^2} dx'dy' \\
&\quad \simeq \frac{h_{i+1}}{h_i} \int_0^1 \int_0^1 \frac{(\tilde{\phi}^h(-x') - \tilde{\phi}^h(y'))^2}{|x' + y'|^2} dx'dy' \\
&\quad \simeq (\phi_i - \phi_{i+1})^2 + (\phi_{i+1} - \phi_{i+2})^2.
\end{aligned}
$$

This completes the proof of our lemma. □

Then we obtain the following theorem.

Theorem 2.3 *Suppose that there exist constants c_1 and c_2 such that, for any $u_i^h \in H_h(\Omega_i)$ with $u_i^h = \phi^h$ on γ, the inequality*

$$
\|\phi^h\|_{H_h^{1/2}(\partial\Omega_i)} \leq c_1 \|u_i^h\|_{H^1(\Omega_i)}
$$

holds, and, for any $\phi^h \in H_h^{1/2}(\partial\Omega_i)$, there exist $u_i^h \in H_h(\Omega_i)$ with $u_i^h = \phi^h$ such that

$$
\|u_i^h\|_{H^1(\Omega_i)} \leq c_2 \|\phi^h\|_{H_h^{1/2}(\partial\Omega_i)}.
$$

Then

$$(S_i\phi, \phi) \simeq \|\phi^h\|^2_{H_h^{1/2}(\partial\Omega_i)}.$$

Here ϕ is the vector representation of ϕ^h.

We have just treated the Laplacian equation. For the general elliptic case, S is more complicated so that it is difficult to find a preconditioner for S. But the following lemma shows that it is enough to find a preconditioner for the Laplacian equation.

Lemma 2.4 *Let* $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ *and* $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$. *Assume that* $A = A^\top \geq 0$, $B = B^\top \geq 0$, $A_{11} > 0$, *and* $B_{11} > 0$. *Let* $S_A = A_{22} - A_{21}A_{11}^{-1}A_{12}$ *and* $S_B = B_{22} - B_{21}B_{11}^{-1}B_{12}$. *If*

$$c_1(Au, u) \leq (Bu, u) \leq c_2(Au, u) \qquad \forall u$$

then

$$c_1(S_A u, u) \leq (S_B u, u) \leq c_2(S_A u, u) \qquad \forall u.$$

Proof. We start with the lower estimate. Indeed,

$$\begin{aligned}
(S_B u_2, u_2) &= \inf_{u_1} \left(B \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) \\
&= \left(B \begin{bmatrix} v_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ u_2 \end{bmatrix} \right) \qquad \text{(for some } v_1\text{)} \\
&\geq c_1 \left(A \begin{bmatrix} v_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ u_2 \end{bmatrix} \right) \\
&\geq c_1 \inf_{u_1} \left(A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) = c_1(S_A u_2, u_2).
\end{aligned}$$

The upper estimate can be proved with the same arguments. ◻

Let

$$Lu = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) + a_0(x)u = f(x), \; x \in \Omega.$$

Assume that

$$(Lu, u) \simeq \|u\|^2_{H^1(\Omega)}.$$

Let S_L and S_1 be Schur complement matrices for L and -Δ, respectively. Lemma 2.4 implies that it is enough to construct a preconditioner for S_1 in place of S_L.

3 Finite Element Trace Theorem

To construct effective preconditioners on interfaces we need the following result about the analytical characterization of finite element traces.

Theorem 3.1 *Let Ω be a bounded domain with the piecewise smooth boundary Γ, and let Ω^h (Ω^h is a polygonal approximation of Ω whose vertex may not lie on Γ) be a shape-regular triangulation of Ω such that*

i) we have

$$\frac{\text{diam } \tau_i}{r_i} \leq c \neq c(h),$$

where r_i denotes the radius of the largest ball inscribed in τ_i,

ii) there exists a mapping $T : \tau_i \to \tilde{\tau}_i$ such that $T(z_i) = \tilde{z}_i$ (z_i and $\tilde{z}_i$ are the vertices of τ_i and $\tilde{\tau}_i$, respectively) and

- $T(\tau_i) = \tilde{\tau}_i$ *is also shape regular,*
- $z_i \in \Gamma^h \Longrightarrow \tilde{z}_i \in \Gamma$ *(The map T moves $z_i \in \Gamma^h$ to $\tilde{z}_i \in \Gamma$.),*
- $\exists c_1, c_2 \neq c(h),\ c_1|z_i - z_j| \leq |\tilde{z}_i - \tilde{z}_j| \leq c_2|z_i - z_j|$.

Then

(1) There exists a constant $c_3 \neq c_3(h)$ such that

$$\|\varphi^h\|_{H_h^{1/2}(\Gamma^h)} \leq c_3\|u^h\|_{H^1(\Omega^h)} \quad \forall u^h \in H_h(\Omega^h) \text{ with } u^h|_{\Gamma^h} = \varphi^h.$$

(2) There exists a constant $c_4 \neq c_4(h)$ such that, for any given $\varphi^h \in H_h(\Gamma^h)$, there exists a finite element function $u^h \in H_h(\Omega^h)$ satisfying the trace condition $u^h = \varphi^h$ on Γ_h and the inequality

$$\|u^h\|_{H^1(\Omega^h)} \leq c_4\|\varphi^h\|_{H^{1/2}(\Gamma^h)}.$$

Remark 3.2 Such mappings T really exist if $\Gamma^h \approx \Gamma$ in $O(h^2)$ accuracy.

In the paper [12] by Korneev (1970), the special finite element function $\tilde{u}^h \in H_h(\widetilde{\Omega}^h)$ on the curvilinear triangulation $\widetilde{\Omega}^h$ was suggested such that $\tilde{u}^h(\tilde{z}_i) = u^h(z_i)$ where $\tilde{u}^h \in H_h(\widetilde{\Omega}^h)$ and $u^h \in H_h(\Omega^h)$. Moreover, the following lemma holds.

Lemma 3.3 *There exist constants c_5 and $c_6 \neq c(h)$ such that*

$$c_5\|\tilde{u}^h\|_{L^2(\tilde{\tau}_i)} \leq \|u^h\|_{L^2(\tau_i)} \leq c_6\|\tilde{u}^h\|_{L^2(\tilde{\tau}_i)},$$
$$c_5|\tilde{u}^h|_{H^1(\tilde{\tau}_i)} \leq |u^h|_{H^1(\tau_i)} \leq c_6|\tilde{u}^h|_{H^1(\tilde{\tau}_i)},$$
$$c_5\|\tilde{\varphi}^h\|_{L^2(\tilde{I}_i)} \leq \|\varphi^h\|_{L^2(I_i)} \leq c_6\|\tilde{\varphi}^h\|_{L^2(\tilde{I}_i)},$$

and

$$c_5 \int_{\tilde{I}_i} \int_{\tilde{I}_j} \frac{(\tilde{\varphi}^h(x) - \tilde{\varphi}^h(y))^2}{|x-y|^2} \, dxdy \leq \int_{I_i} \int_{I_j} \frac{(\varphi^h(x) - \varphi^h(y))^2}{|x-y|^2} \, dxdy \leq c_6 \int_{\tilde{I}_i} \int_{\tilde{I}_j} \frac{(\tilde{\varphi}^h(x) - \tilde{\varphi}^h(y))^2}{|x-y|^2} \, dxdy,$$

where $\Gamma^h = \bigcup_i I_i$ *and* $\Gamma = \bigcup_i \tilde{I}_i$.

The proof of Lemma 3.3 is based on the result from [12]. Now, we are able to prove Theorem 3.1.

Proof (Existence of c_3*).* There obviously exists a constant c_7 such that, for any given $u^h \in H_h(\Omega^h)$, there is a $\tilde{u}^h \in H^1(\tilde{\Omega}^h)$ satisfying the inequality $\|\tilde{u}^h\|_{H^1(\tilde{\Omega}^h)} \leq c_7 \|u^h\|_{H^1(\Omega^h)}$. Setting $\tilde{\varphi}^h = \tilde{u}^h|_\Gamma \in H_h(\Gamma)$, we define $\varphi^h \in H_h(\Gamma_h)$ as a linear combinations of vertex values of $\tilde{\varphi}^h$. Now, we get the inequality

$$\|\tilde{\varphi}^h\|_{H^{1/2}(\Gamma)} \leq c_8 \|\tilde{u}^h\|_{H^1(\Omega)}$$

from the usual trace theorem. By Lemma 3.3, it follows that

$$\|\varphi^h\|_{H_h^{1/2}(\Gamma^h)} \leq c_9 \|\tilde{\varphi}^h\|_{H^{1/2}(\Gamma)}.$$

We remark that this is immediate in the case $\Omega^h = \overline{\Omega}$. □

Proof (Existence of c_4*).* For a given $\varphi^h \in H_h(\Gamma^h)$, let $\tilde{\varphi}^h \in H_h(\Gamma)$ be such that $\tilde{\varphi}^h(\tilde{z}_i) = \varphi^h(z_i)$. Then we have by Lemma 3.3

$$\|\tilde{\varphi}^h\|_{H^{1/2}(\Gamma)} \leq c \|\varphi^h\|_{H_h^{1/2}(\Gamma^h)}.$$

By the inverse trace theorem, there exists $u \in H^1(\Omega)$ such that $u|_\Gamma = \tilde{\varphi}^h$ and $\|u\|_{H^1(\Omega)} \leq c \|\tilde{\varphi}^h\|_{H^{1/2}(\Gamma)}$. But $u \notin H_h(\Omega)$. How can we construct $\tilde{u}^h \in H_h(\Omega)$? It is enough to have values at $\tilde{z}_i$. Let

$$\tilde{u}^h(\tilde{z}_i) = \begin{cases} \tilde{\varphi}^h(\tilde{z}_i), & \text{if } \tilde{z}_i \in \Gamma, \\ \dfrac{1}{\pi r_i} \displaystyle\int_{B(\tilde{z}_i, r_i)} u(x) \, dx, & \text{otherwise,} \end{cases}$$

where r_i is the radius of the largest ball $B(\tilde{z}_i, r_i)$ inscribed in the union of all elements sharing the vertex $\tilde{z}_i$ which is denoted by K_i. Then we take $u^h \in H_h(\Omega^h)$ with $u^h(z_i) = \tilde{u}^h(\tilde{z}_i)$. By Lemma 3.3 it follows that $\|u^h\|_{H^1(\Omega^h)} \leq c \|\tilde{u}^h\|_{H^1(\Omega)}$.

It remains to show that $\|\tilde{u}^h\|_{H^1(\Omega)} \leq c \|\tilde{\varphi}^h\|_{H^{1/2}(\Gamma)}$. We note that $\varphi^h \to \tilde{\varphi}^h \to u \to \tilde{u}^h \to u^h$. By Friedrich's inequality, we obtain

$$\|\tilde{u}^h\|_{L^2(\Omega)} \leq c(|\tilde{u}^h|_{H^1(\Omega)} + \|\tilde{\varphi}^h\|_{L^2(\Gamma)}),$$

and since $\|\tilde{\varphi}^h\|_{L^2(\Gamma)} \leq C\|\varphi\|_{H^{1/2}(\Gamma)}$ it is enough to estimate $|\tilde{u}^h|_{H^1(\Omega)}$. We note that

$$|\tilde{u}^h|^2_{H^1(\Omega)} \leq c \sum_{l_i \in \tilde{\Omega}^h} (\tilde{u}^h(\tilde{z}_{i_1}) - \tilde{u}^h(\tilde{z}_{i_2}))^2,$$

where $\tilde{z}_{i_1}$ and $\tilde{z}_{i_2}$ are the vertices of the edge l_i. Now we consider the following three cases separately:

- Case 1) $\tilde{z}_{i_1}, \tilde{z}_{i_2} \in \Gamma$. In this case, we immediately get

$$\begin{aligned}\sum (\tilde{u}^h(\tilde{z}_{i_1}) - \tilde{u}^h(\tilde{z}_{i_2}))^2 &= \sum (\tilde{\varphi}^h(\tilde{z}_{i_1}) - \tilde{\varphi}^h(\tilde{z}_{i_2}))^2 \\ &\leq \sum_{\tilde{z}_i} \sum_{\tilde{z}_j} \frac{(\tilde{\varphi}^h(\tilde{z}_i) - \tilde{\varphi}^h(\tilde{z}_i))^2}{|z_i - z_j|^2} h_i h_j \leq c\,|\tilde{\varphi}^h|^2_{H^{1/2}(\Gamma)}.\end{aligned}$$

- Case 2) $\tilde{z}_{i_1}, \tilde{z}_{i_2} \in \Omega$. Here, we use the following result.

Lemma 3.4 *Let $0 < h_1 \leq h_2$. Then we have the estimate*

$$\left(\frac{1}{\pi h_2^2} \int_{B(0,h_2)} u(x)\,dx - \frac{1}{\pi h_1^2} \int_{B(0,h_1)} u(x)\,dx \right)^2 \leq \frac{h_2}{\pi h_1} |u|^2_{H^1(B(0,h_2))}$$

that is valid for all $u \in H^1(B(0,h_2))$.

Proof. Let (r,θ) be the radial coordinate system given by the relation

$$x = (x_1, x_2) = (r\cos\theta, r\sin\theta).$$

Then we obtain

$$\begin{aligned}
&\left(\frac{1}{\pi h_2^2}\int_{B(0,h_2)} u(x)\,dx - \frac{1}{\pi h_1^2}\int_{B(0,h_1)} u(x)\,dx\right)^2 \\
&= \left(\frac{1}{\pi h_2^2}\int_0^{h_2}\int_0^{2\pi} u(r,\theta)\,rd\theta dr - \frac{1}{\pi h_1^2}\int_0^{h_1}\int_0^{2\pi} u(r,\theta)\,rd\theta dr\right)^2 \\
&= \left(\frac{1}{\pi h_2^2}\int_0^{h_2}\int_0^{2\pi} (u(r,\theta)-u(r/a,\theta))\,rd\theta dr\right)^2 \qquad (a = h_2/h_1 \geq 1) \\
&= \frac{1}{\pi^2 h_2^4}\left(\int_0^{h_2}\int_0^{2\pi}\left[\int_{\frac{r}{a}}^{r} r^{1/2}\frac{\partial u(t,\theta)}{\partial t}\,dt\right] r^{1/2} d\theta dr\right)^2 \\
&\overset{\text{C.B.}}{\leq} \frac{1}{\pi h_2}\int_0^{h_2}\int_0^{2\pi}\int_{\frac{r}{a}}^{r}\left(\frac{\partial u(t,\theta)}{\partial t}\right)^2 r\,dtd\theta dr \\
&\overset{r\leq at}{\leq} \frac{a}{\pi h_2}\int_0^{h_2}\int_0^{2\pi}\int_{\frac{r}{a}}^{r}\left(\frac{\partial u(t,\theta)}{\partial t}\right)^2 t\,dtd\theta dr \\
&\leq \frac{a}{\pi h_2}\int_0^{h_2}\int_0^{2\pi}\int_0^{h_2}\left(\frac{\partial u(t,\theta)}{\partial t}\right)^2 t\,dtd\theta dr \\
&= \frac{a}{\pi}\int_0^{2\pi}\int_0^{h_2}\left(\frac{\partial u(t,\theta)}{\partial t}\right)^2 t\,dtd\theta \\
&\leq \frac{a}{\pi}|u|^2_{H^1(B(0,h_2))}.
\end{aligned}$$

The last inequality follows from the fact that $(\frac{\partial u}{\partial r})^2 \leq (\frac{\partial u}{\partial x_1})^2 + (\frac{\partial u}{\partial x_2})^2$. This finishes the proof of Lemma 3.4. □

Now, we can continue the proof with case 2).

Can we find a constant c_4 such that for all $\phi_h \in H_h(\Gamma^h)$ there exists a finite element function $u^h \in H_h(\Omega^h)$ satisfying the trace condition $u^h(x) = \phi^h(x),\ x \in \Gamma^h$ and the inequality $\|u^h\|_{H^1(\Omega^h)} \leq c_4|\phi^h|_{H_h^{1/2}(\Gamma^h)}$? Using the construction $\phi^h \to \tilde{\phi}^h \in \tilde{H}_h(\Gamma) \to u \in H^1(\Omega) \to \tilde{u}^h \in \tilde{H}_h(\Omega)$ with

$$\tilde{u}^h(\tilde{z}_i) = \frac{1}{\pi r_i^2}\int_{B(\tilde{z}_i,r_i)} u(x)dx, \tag{3.1}$$

Note 2: May we delete r: ?

we can proceed as follows. There are two cases:

1) $\tilde{z}_i, \tilde{z}_j \in \Gamma$

2) $\tilde{z}_i, \tilde{z}_j \in \Omega$

Let r denote the radius satisfying the following inclusion:

$$r\colon B(x,\sqrt{2}r) \subset K_{i_1} \cup K_{i_2}, \quad x \in l_i.$$

Now we estimate $(\tilde{u}^h(\tilde{z}_{i_2}) - \tilde{u}^h(\tilde{z}_{i_1}))^2$ as follows:

$$(\tilde{u}^h(\tilde{z}_{i_2}) - \tilde{u}^h(\tilde{z}_{i_1}))^2 \leq 3\left(\left(\tilde{u}^h(\tilde{z}_{i_2}) - \frac{1}{\pi r^2}\int_{B(\tilde{z}_{i_2},r)} u(x)dx\right)^2 + \left(\frac{1}{\pi r^2}\int_{B(\tilde{z}_{i_1},r)} u(x)dx - \tilde{u}^h(\tilde{z}_{i_1})\right)^2 + \left(\frac{1}{\pi r^2}\int_{B(\tilde{z}_{i_2},r)} u(x)dx - \frac{1}{\pi r^2}\int_{B(\tilde{z}_{i_1},r)} u(x)dx\right)^2\right).$$

For the first two terms we can use Lemma 3.4. Let us now estimate the third term:

$$\begin{aligned}
&\left(\frac{1}{\pi r^2}\int_{B(\tilde{z}_{i_2},r)} u(x)dx - \frac{1}{\pi r^2}\int_{B(\tilde{z}_{i_1},r)} u(x)dx\right)^2 \\
&= \frac{1}{\pi^2 r^4}\left(\int_{B(\tilde{z}_{i_1},r)} (u(x+y) - u(x))\cdot 1dx\right)^2 \\
&\leq \frac{1}{\pi r^2}\int_{B(\tilde{z}_{i_1},r)} (u(x+y) - u(x))^2 dx \\
&\leq \frac{1}{\pi r^2}\int_{-r}^{r}\int_{-r}^{r} (u(s+h,t) - u(s,t))^2 dsdt \\
&\leq \frac{1}{\pi r^2}\int_{-r}^{r}\int_{-r}^{r}\int_{s}^{s+h}\left(\frac{\partial u(\xi,t)}{\partial \xi}\right)^2 d\xi dsdt \\
&\leq \frac{1}{\pi r^2}\int_{-r}^{r}\int_{-r}^{r}\int_{-r}^{r+h}\left(\frac{\partial u(\xi,t)}{\partial \xi}\right)^2 d\xi dsdt \\
&= \frac{2h}{\pi r^2}\int_{-r}^{r}\int_{-r}^{r+h}\left(\frac{\partial u(\xi,t)}{\partial \xi}\right)^2 d\xi dt \\
&\leq \frac{2h}{\pi r^2}|u|_{H^1(K_{i_1}\cup K_{i_2})}.
\end{aligned}$$

• Case 3) $\tilde{z}_{i_1} \in \Gamma$, $\tilde{z}_{i_2} \in \Omega$. Next, let $\tilde{z}_{i_1} = (0,0)$, $\tilde{z}_{i_1}^{+} = (h_1, 0)$, $\tilde{z}_{i_1}^{-} = (-h_2, 0)$, $\tilde{z}_{i_2} = (0, h_3)$, and r: $B(\tilde{z}_i, r) \subset S$, where $S = \{(s,h)| - h_2 \leq s \leq h_1, 0 \leq h \leq 2h_3\}$. Then, we have

Note 3: May we write: and r with $B(\ldots$

$$(\tilde{u}^h(\tilde{z}_{i_2}) - \tilde{u}^h(\tilde{z}_{i_1}))^2 \leq 2\left(\tilde{u}^h(\tilde{z}_{i_2}) - \frac{1}{\pi r^2}\int_{B_(\tilde{z}_{i_2},r)} u(x)dx\right)^2 + \left(\frac{1}{\pi r^2}\int_{B_(\tilde{z}_{i_2},r)} u(x)dx - \tilde{u}^h(\tilde{z}_{i_1})\right)^2.$$

The second term can be estimated as follows:

$$\begin{aligned}
&\frac{1}{\pi^2 r^4}\left(\int_{B(\tilde z_{i_1},r)}(u(x)-\tilde u^h(\tilde z_{i_2}))dx\right)^2 \overset{\text{C.B.}}{\le} \frac{1}{\pi r^2}\int_{B(\tilde z_{i_1},r)}(u(x)-\tilde u^h(\tilde z_{i_2}))^2dx\\
\le\;& \frac{1}{\pi r^2}\int_{-h_2}^{h_1}\int_0^{2h_3}(u(s,t)-\phi^h(0))^2dtds\\
\le\;& \frac{2}{\pi r^2}\left(\int_{-h_2}^{h_1}\int_0^{2h_3}(u(s,t)-\phi^h(s))^2dtds+\int_{-h_2}^{h_1}\int_0^{2h_3}(\phi(s)-\phi^h(0))^2dtds\right)\\
\le\;& \frac{2}{\pi r^2}\left(\int_{-h_2}^{h_1}\int_0^{2h_3}\left(\int_0^t\frac{\partial u(s,\xi)}{\partial\xi}d\xi\right)^2dtds\right.\\
&\left.+\,2\left(\int_0^{h_1}\left(\tilde\phi(s)-\tilde\phi^h(0)\right)^2ds+\int_{-h_2}^0\left(\tilde\phi(s)-\tilde\phi^h(0)\right)^2ds\right)\right)\\
\le\;& C\left(|u|^2_{H^1(S)}+(\tilde\phi^h(z^+_{i_1})-\tilde\phi^h(z_{i_1}))^2+(\tilde\phi^h(z^-_{i_1})-\tilde\phi^h(z_{i_1})^2\right)^2.
\end{aligned}$$

Finally, we have

$$\|u^h\|_{H^1(\Omega^h)}\le C_4\|\phi^h\|_{H_h^{1/2}(\Gamma^h)}.$$

□

In the following we need Sobolev's norm equivalence theorem, the proof of which can be found in [43].

Theorem 3.5 *Let* $l\colon H^1(\Omega)\to R$ *be a linear bounded functional. If* $l(c)=0$ *for some constant* c *yields* $c=0$*, then* $\|u\|_{H^1(\Omega)}\approx|u|_{H^1(\Omega)}+|l(u)|$.

For instance, this theorem immediately yields the well-known Poincaré inequality

$$\|u\|^2_{L^2(\Omega)}\le C\left(|u|^2_{H^1(\Omega)}+\left(\int_\Omega u(x)dx\right)^2\right).$$

If $\int_\Omega u\,dx=0$, then we have the usual Poincaré–Friedrich's inequality.

Lemma 3.6 (Poincaré inequality in $H^{1/2}(\Gamma)$)

$$\int_\Gamma\phi^2(x)dx\le C\left(\int_\Gamma\int_\Gamma\frac{|\phi(x)-\phi(y)|^2}{|x-y|^2}dxdy+\left(\int_\Gamma\phi(x)dx\right)^2\right). \tag{3.2}$$

Proof. Let $x,y\in\Gamma$ and $x\ne y$ then

$$(\phi(x)-\phi(y))^2\le C_0\frac{(\phi(x)-\phi(y))^2}{|x-y|^2},$$

where $C_0=\operatorname{diam}\Omega$. Thus, we have

$$\int_\Gamma\int_\Gamma(\phi(x)-\phi(y))^2dxdy\le C_0\int_\Gamma\int_\Gamma\frac{(\phi(x)-\phi(y))^2}{|x-y|^2}\,dydy.$$

Substituting

$$\begin{aligned}
&\int_\Gamma \int_\Gamma (\phi(x) - \phi(y))^2 dx dy \\
&= \int_\Gamma \int_\Gamma \phi(x)^2 dx dy - 2 \int_\Gamma \int_\Gamma \phi(x)\phi(y)^2 dx dy + \int_\Gamma \int_\Gamma \phi(y)^2 dx dy \\
&= 2 \cdot \text{meas}(\Gamma) \int_\Gamma \phi^2(x) dx - 2 \left(\int_\Gamma \phi(x) dx \right)^2
\end{aligned}$$

into above equation, we arrive at (3.2). □

Theorem 3.7 (Trace theorem with semi-norm) *There are two positive constants C_1 and C_2 such that*

1) for all $u \in H^1(\Omega)$ its trace ϕ on Γ satisfies the inequality

$$|\phi|_{H^{1/2}(\Gamma)} \le C_1 |u|_{H^1(\Omega)},$$

and

2) for all $\phi \in H^{1/2}(\Gamma)$ there exists a function $u \in H^1(\Omega)$ such that $u = \phi$ on Γ and

$$|u|_{H^1(\Omega)} \le C_2 |\phi|_{H^{1/2}(\Gamma)}.$$

Proof. Let $u \in H^1(\Omega)$. Then, the function u can be split into two parts as follows:

$$u = u_0 + u_1, \quad u_0 = \text{ constant } = \frac{1}{\text{meas}(\Omega)} \int_\Omega u d\Omega, \quad \int_\Omega u_1 d\Omega = 0.$$

Now we split $\phi = \phi_0 + \phi_1$ into the traces $\phi_0 = u_0|_\Gamma$ and $\phi_1 = u_1|_\Gamma$ of the functions u_0 and u_1 on Γ. Then we have

$$|\phi|_{H^{1/2}(\Gamma)} = |\phi_1|_{H^{1/2}(\Gamma)} \le C_3 \|u_1\|_{H^1(\Omega)} \le C_4 |u_1|_{H^1(\Omega)} = C_4 |u|_{H^1(\Omega)}.$$

This completes the proof of the first statement of the theorem.

The second statement can be proved as follows: Let $\phi \in H^{1/2}(\Gamma)$ be decomposed as

$$\phi = \phi_0 + \phi_1, \quad \phi_0 = \text{ constant } = u_0, \quad \int_\Gamma \phi_1 d\Gamma = 0.$$

By the standard trace theorem, there exists u_1 such that $u_1(x) = \phi_1(x)$ and

$$\|u_1\|_{H^1(\Omega)} \le C_5 \|\phi_1\|_{H^{1/2}(\Gamma)}.$$

Set $u = u_0 + u_1$. Then $u(x) = \phi(x), x \in \Gamma$ and

$$|u|^2_{H^1(\Omega)} = |u_1|^2_{H^1(\Omega)} \le C_5 \|\phi_1\|^2_{H^{1/2}(\Gamma)} \le C_6 |\phi|^2_{H^{1/2}(\Gamma)}$$

where the last estimate follows from the Poincaré inequality . □

Remark 3.8 We have the same theorem for the finite element space because the FEM space contains the constant function.

Note that the definition of $\|\phi\|_{H^{1/2}(\Gamma)}$ is very complicated.
If $\phi \in H^1(-1,1)$ then we have

$$\|\phi\|^2_{H^1(-1,1)} = \|\phi\|^2_{H^1(-1,0)} + \|\phi\|^2_{H^1(0,1)}.$$

Note 4:
Or: when the norm in H^α is additive with respect to the domain's measure. (?)

This kind of identity is only true for H^α with $0 < \alpha \leq 1$ and $\alpha \neq 1/2$, when norm in H^α is additive with to respect of the measure of domain. In general cases we use the norm

$$\|\phi\|^2_{H^\alpha} = \|\phi\|^2_{L^2} + \int_{-1}^{1}\int_{-1}^{1} \frac{(\phi(x)-\phi(y))^2}{|x-y|^{1+2\alpha}}\,dxdy$$

and the following lemma.

Lemma 3.9 *There exist positive constants c_1 and c_2 such that*

$$\begin{aligned} c_1\|\phi\|^2_{H^{1/2}(-1,1)} &\leq \|\phi\|^2_{H^{1/2}(-1,0)} + \|\phi\|^2_{H^{1/2}(0,1)} + \int_0^1 \frac{(\phi(x)-\phi(-x))^2}{x}\,dx \\ &\leq c_2\|\phi\|^2_{H^{1/2}(-1,1)} \end{aligned}$$

Proof. We have

$$\int_0^1 \frac{dy}{(x+y)^2} = \int_x^{1+x} \frac{dt}{t^2} = \frac{1}{x(1+x)}.$$

Thus,

$$\frac{1}{2x} \leq \int_0^1 \frac{dy}{(x+y)^2} \leq \frac{1}{x}, \quad x \in (0,1].$$

Then we can estimate the third term $I(\phi)$ in the first inequality as follows:

$$\begin{aligned} I(\phi) &= \int_0^1 \frac{(\phi(x)-\phi(-x))^2}{x}\,dx \\ &\leq \int_0^1\int_0^1 \frac{(\phi(x)-\phi(-x))^2}{(x+y)^2}\,dydx \\ &\leq 4\int_0^1\int_0^1 \frac{(\phi(x)-\phi(y))^2}{(x+y)^2}\,dydx + 4\int_0^1\int_0^1 \frac{(\phi(y)-\phi(-x))^2}{(x+y)^2}\,dydx \\ &\leq 4(|\phi|^2_{H^{1/2}(0,1)} + |\phi|^2_{H^{1/2}(-1,1)}). \end{aligned}$$

Thus, the second inequality is proved. For the first inequality, we only need to consider the semi norm. First of all we have the representations

$$|\phi|^2_{H^{1/2}(0,1)} = \int_0^1\int_0^1 \frac{(\phi(x)-\phi(y))^2}{(x-y)^2}\,dydx$$

and

$$|\phi|^2_{H^{1/2}(-1,1)} = \int_0^1\int_0^1 \frac{(\phi(x)-\phi(y))^2}{(x-y)^2}\,dydx + \int_0^1\int_{-1}^0 + \int_{-1}^0\int_0^1 + \int_{-1}^0\int_{-1}^0$$
$$= I + II + III + IV.$$

Since $I = |\phi|^2_{H^{1/2}(0,1)}$ and $IV = |\phi|^2_{H^{1/2}(-1,0)}$, we have to consider only $II = III$. So, we estimate only one of them:

Note 5: May we delete the number (3.3) and place the last line behind the one before?

$$\int_{-1}^0\int_0^1 \frac{(\phi(x)-\phi(y))^2}{(x-y)^2}\,dydx$$
$$= 2\left(\int_{-1}^0\int_0^1 \frac{(\phi(x)-\phi(-x))^2}{(x-y)^2}dydx + \int_{-1}^0\int_0^1 \frac{(\phi(-x)-\phi(y))^2}{(x-y)^2}\right)dydx$$
$$\le 2\int_0^1 (\phi(-x)-\phi(y))^2 \int_0^1 \frac{dy}{(x+y)^2} \tag{3.3}$$
$$+ \int_{-1}^0\int_{-1}^0 \frac{(\phi(x)-\phi(y))^2}{(x+y)^2}\,dydx.$$

Here we have used the change of variables $(y \to -y')$, $(x \to -x')$. Now, we use the estimates of the integral

$$\frac{1}{2x} \le \int_0^1 \frac{dy}{(x+y)^2} \le \frac{1}{x}$$

to see that the third term is less than $|\phi|^2_{H^{1/2}(-1,0)}$. Then $I + II + III + IV$ can be estimated by

Note 6: Or: Finish the formula with fulls stop and start next line "That proves"

$$2(I(\phi) + |\phi|^2_{H^{1/2}(-1,0)})$$

that proves the fist inequality in Lemma 3.9. □

Now divide the boundary (like a circle) by two points a and b on Γ, and the left-hand side is called Γ_1. Let us consider

$$H_{00}^{1/2}(\Gamma_1).$$

Let us assume that ϕ is equal to zero on $\Gamma_0 = \Gamma\backslash\Gamma_1$ and equivalent to the harmonic extension into the interior, i.e.,

$$\|\phi\|^2_{H^{1/2}(\Gamma)} \approx \|\phi\|^2_{H^{1/2}(\Gamma_1)} + \|\phi\|^2_{H^{1/2}(\Gamma_0)}(=0) + \int_{\Gamma_1}\frac{\phi^2}{|x-a|^2} + \int_{\Gamma_1}\frac{\phi^2}{|x-b|^2}.$$

With this motivation, we define

$$\|\phi\|^2_{H_{00}^{1/2}(\Gamma_1)} = \|\phi\|^2_{H^{1/2}(\Gamma_1)} + \int_{\Gamma_1}\frac{\phi^2}{|x-a|^2} + \int_{\Gamma_1}\frac{\phi^2}{|x-b|^2}. \tag{3.4}$$

Similarly, we introduce

Note 7: Really $\overset{00}{H}{}_h^{1/2}(0,1)$?

$$\|\phi\|_{\overset{00}{H}{}^{1/2}(0,1)} = \|\phi\|^1_{H^{1/2}(0,1)} + \int_0^1 \frac{\phi^2}{x(1-x)}. \tag{3.5}$$

Meanwhile a function in $H^{1/2}(0,1)$ does not have anything to do with the value outside $(0,1)$.

Now, let us consider the FEM case. Let Ω be triangularized by Ω^h. Some part of its boundary is denoted by Γ_1^h some other by Γ_0^h. We now define the discrete counter part

$$\|\phi^h\|_{\overset{00}{H}{}^{1/2}_h(\Gamma_1^h)} = \|\phi^h\|^1_{H_h^{1/2}(\Gamma_1^h)} + \sum_{z_i \in \Gamma_1^h} \frac{(\phi^h(z_i))^2}{|z_i - a|} h_i + \sum_{z_i \in \Gamma_1^h} \frac{(\phi^h(z_i))^2}{|z_i - b|} h_i$$

of the norm. The second two terms correspond to an analog in the space $H^{1/2}$.

Let $\overset{0}{H}_h(\Gamma_1^h) = \{\phi^h \in H_h^{1/2}(\Gamma_1^h) | \ \phi^h(a) = \phi^h(b) = 0\}$ and $\phi^h \to \phi \in \mathbb{R}^n$. Then we have the following equivalences:

$$(S\phi, \phi) \approx \|\phi^h\|^2_{H^{1/2}(\Gamma^h)} \approx \|\phi^h\|^2_{\overset{00}{H}{}^{1/2}(\Gamma_1^h)} \approx \|\tilde{\phi}^h\|^2_{\overset{00}{H}{}^{1/2}(I)}.$$

Note 8: Or: a/the straightened boundary

Note 9: Or: grid, gives

Here I is straightened boundary. Now, setting $\phi^h(z_i) = \tilde{\phi}(\tilde{z}_i)$ by mapping, extending it into the unit square and considering it on a uniform grid give

$$\|\tilde{\phi}^h\|_{\overset{00}{H}{}^{1/2}(I)} \approx (\tilde{S}\phi, \phi).$$

Finally, we have

$$(S\phi, \phi) \approx (\tilde{S}\phi, \phi).$$

Hence a preconditioner for $\tilde{S}$ suffices for the original problem. In summary, the Schur complement S is equivalent to the interface norm which is in turn equivalent to Schur complement $\tilde{S}$. On a good domain, the Schur complement $\tilde{S}$ can be found analytically.

A detailed study on the space $H_{00}^{1/2}(\Gamma_1)$

Let us start with a review on the Schur complement norm. First we recall that

$$(S\varphi, \varphi) = \inf_{w^h|_\Gamma = \varphi^h} \|w^h\|^2_{H^1(\Omega)} = \|u^h\|^2_{H^1(\Omega)}, \tag{3.6}$$

where u^h satisfying $u^h|_\Gamma = \varphi$ is the minimizer. Then, it is clear that

$$\|\varphi^h\|^2_{H^{1/2}(\Gamma)} \le C_3 \, \|u^h\|^2_{H^1(\Omega)} = C_3 \, (S\varphi, \varphi).$$

For φ^h, there exists a constant C_4 such that

$$\|v^h\|^2_{H^1(\Omega)} \le C_4 \|\varphi^h\|^2_{H^{1/2}(\Gamma)}.$$

Thus, we have

$$(S\varphi, \varphi) = \inf_{w^h|_\Gamma = \varphi^h} \|w^h\|^2_{H^1(\Omega)} = \|u^h\|^2_{H^1(\Omega)} \le \|v^h\|^2_{H^1(\Omega)} \le C_4 \|\varphi^h\|^2_{H^{1/2}(\Gamma)}.$$

We see that the Schur complement norm is equivalent to $H^{1/2}(\Gamma)$ norm. Let $\dot{C}^{\infty}(0,1)$ be the subspace of $C^{\infty}(0,1)$ with compact support. Then it is well known that

$$\overline{(C^{\infty}(0,1))}_{L^2} = L^2(0,1) \text{ and } \overline{(\dot{C}^{\infty}(0,1))}_{L^2} = L^2(0,1).$$

However, in the H^1 case, we have the closure relations

$$\overline{(C^{\infty}(0,1))}_{H^1(0,1)} = H^1(0,1) \text{ and } \overline{(\dot{C}^{\infty}(0,1))}_{H^1(0,1)} = H^1_0(0,1).$$

The definition of H^{α} for $\alpha \leq 1/2$ follows from the L^2 case and for $\alpha > 1/2$ follows from the H^1 case:

$$\|\varphi\|^2_{H^{\alpha}(0,1)} = \|\varphi\|^2_{L^2(0,1)} + \int_0^1 \int_0^1 \frac{(\varphi(x) - \varphi(y))^2}{|x-y|^{1+2\alpha}}\, dxdy.$$

If $\alpha \leq 1/2$ then

$$\overline{(C^{\infty}(0,1))}_{H^{\alpha}} = H^{\alpha}(0,1) \text{ and } \overline{(\dot{C}^{\infty}(0,1))} = H^{\alpha}(0,1). \tag{3.7}$$

If $\alpha < 1/2$ then, for $u \in H^{\alpha}(0,1)$, its extension by zero outside $(0,1)$ belongs to $H^{\alpha}(-1,2)$ like in the case of the L^2 space. However, for $\alpha = 1/2$, a function in $H^{1/2}(0,1)$ cannot be extended by zero (note that $H_0^{1/2} = H^{1/2}(0,1)$). If $\alpha > 1/2$ then

$$\overline{(C^{\infty}(0,1))}_{H^{\alpha}} = H^{\alpha}(0,1) \text{ and } \overline{(\dot{C}^{\infty}(0,1))}_{H^{\alpha}} = H^{\alpha}_0(0,1).$$

Let $\alpha = 1/2$. If we extend $\dot{C}^{\infty}(0,1)$ by the norm $\|\cdot\|_{H^{1/2}_{00}}$, then we obtain $H^{1/2}_{00}(0,1)$ and we can extend the function in $H^{1/2}_{00}(0,1)$ to a function in $H^{1/2}(-1,2)$ by zero. Hence, we have the proper inclusion

$$H^{1/2}(0,1) \supsetneq H^{1/2}_{00}(0,1).$$

We note that

$$\|\varphi\|^2_{H^{1/2}(-1,1)} \approx \|\varphi\|^2_{H^{1/2}(-1,0)} + \|\varphi\|^2_{H^{1/2}(0,1)} + \int_0^1 \frac{(\varphi(x) - \varphi(-x))^2}{x}\, dx.$$

For $\varphi \in H^{1/2}_{00}(0,1)$, we have

$$\|\varphi\|^2_{H^{1/2}_{00}(0,1)} = \|\varphi\|^2_{H^{1/2}(0,1)} + \int_0^1 \frac{(\varphi(x))^2}{x(1-x)}\, dx.$$

Hence, $\varphi \to 0$ as x tends to 0 and 1. Let us define the function $\tilde{\varphi} \in H^{1/2}(-1,2)$ by the formula

$$\tilde{\varphi}(x) = \begin{cases} 0 & x \in (-1,0), \\ \varphi(x) & x \in (0,1), \\ 0 & x \in (1,2). \end{cases}$$

Then we obtain the relations

$$\begin{aligned}
&\|\tilde{\varphi}\|^2_{H^{1/2}(-1,2)} \\
\approx\ & \|\varphi\|^2_{H^{1/2}(0,1)} + \int_0^1 \frac{(\tilde{\varphi}(x) - \tilde{\varphi}(-x))^2}{x}\,dx + \int_0^1 \frac{(\tilde{\varphi}(x) - \tilde{\varphi}(2-x))^2}{1-x}\,dx \\
=\ & \|\varphi\|^2_{H^{1/2}(0,1)} + \int_0^1 \frac{\varphi(x)^2}{x} + \frac{\varphi(x)^2}{1-x}\,dx \\
\approx\ & \|\varphi\|^2_{H^{1/2}(0,1)} + \int_0^1 \frac{\varphi(x)^2}{x(1-x)}\,dx \\
\approx\ & \|\varphi\|^2_{H^{1/2}_{00}}(0,1)
\end{aligned}$$

by simple calculations. In the first equivalence, we omitted both $\|\tilde{\varphi}\|^2_{H^{1/2}(-1,0)}$ and $\|\tilde{\varphi}\|^2_{H^{1/2}(1,2)}$ because they are zero by extension.

If $\alpha > 1/2$ then we have $H^\alpha \hookrightarrow C^0(0,1)$, i.e. if $\varphi \in H^\alpha(0,1)$, $\alpha > 1/2$, then $\lim_{x\to x_0} \varphi(x) = \varphi(x_0)$.

Example 3.10 Let the boundary Γ of Ω be divided by three points a, b and c into three pieces Γ_1, Γ_0 and $\tilde{\Gamma}_1$. Now let us consider the minimization problem

$$\inf_{w\in H^1(\Omega), w|_{\Gamma_1}=\varphi, w|_{\Gamma_0}=0} \|w\|^2_{H^1(\Omega)}.$$

The above minimization problem is obviously equivalent to the mixed boundary value problem

$$\begin{cases} -\triangle w + w = 0, \\ w|_{\Gamma_1} = \varphi, \\ w|_{\Gamma_0} = 0, \\ \dfrac{\partial w}{\partial n}|_{\tilde{\Gamma}_1} = 0. \end{cases}$$

The correct norm in $\check{H}^{1/2}(\Gamma_1)$ is now given by the relation

$$\|\varphi\|^2_{\check{H}^{1/2}(\Gamma_1)} = \|\varphi\|^2_{H^{1/2}(\Gamma_1)} + \int_{\Gamma_1} \frac{(\varphi(x))^2}{|x-a|}\,dx.$$

It is like in $H^{1/2}_{00}(\Gamma_1)$, but it is only an one-side norm because the integral near the point b is missing. For the FEM case, we use the following discrete norm

$$\|\varphi^h\|^2_{\check{H}^{1/2}_h(\Gamma_1)} = \|\varphi^h\|^2_{H^{1/2}_h(\Gamma_1)} + \sum_{z_i\in\Gamma_1} \frac{(\varphi^h(z_i))^2}{|z_i - a|} h_i.$$

4 Domain Decomposition Method: The Strip Case

In this section, the decomposition of Ω into the subdomains Ω_i does not have any cross point. This constellation is called strip case which is characterized by a decomposition of the form

$$\bar{\Omega} = \bigcup_{i=1}^{n} \bar{\Omega}_i, \quad \gamma = \bigcup_{i=1}^{n} \partial\Omega_i \backslash \Gamma = \bigcup_{i=1}^{n-1} \gamma_i, \text{ with } \gamma_i \cap \gamma_j = \emptyset \text{ for } i \neq j,$$

where γ_i is an interface between the subdomains. We again consider the Dirichlet problem for the Poisson equation as model problem:

$$-\triangle u = f(x), \quad x \in \Omega,$$

$$u(x) = 0, \quad x \in \Gamma.$$

Then the finite element discretization yields the following system of linear algebraic equations

$$Au = f$$

which can be rewritten in block form as follows

$$\begin{pmatrix} A_0 & A_{01} & \cdot & \cdot & \cdot & A_{0n} \\ A_{10} & A_1 & & & & \\ \cdot & & \cdot & 0 & & \\ \cdot & & & \cdot & & \\ \cdot & & 0 & & \cdot & \\ A_{n0} & & & & & A_n \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{pmatrix}.$$

Eliminating the vector u_i from the equation

$$A_{i0}u_0 + A_i u_i = f_i$$

and substituting

$$u_i = -A_i^{-1}A_{i0}u_0 + A_i^{-1}f_i$$

into the first (block) equation, we arrive at the Schur complement equations

$$S\varphi = \psi, \tag{4.1}$$

where $S = A_0 - \sum_{i=1}^{n} A_{0i}A_i^{-1}A_{i0}$, $\varphi = u_0$ and $f_0 - \sum_{i=1}^{n} A_{0i}A_i^{-1}f_i$. The Schur complement system (4.1) can be solved, for instance, by the Richardson iteration

$$\varphi^{k+1} = \varphi^k - \tau_k\Sigma^{-1}(S\varphi^k - \psi),$$

where Σ is a suitable preconditioner satisfying the spectral equivalence inequalities

$$c_1(\Sigma\varphi, \varphi) \leq (S\varphi, \varphi) \leq C_2(\Sigma\varphi, \varphi) \tag{4.2}$$

for all vectors φ, and $\hat{\tau}_k$ are appropriately chosen iteration parameters.

Now we arrange the vector u_0 in the form

$$u_0 = \begin{pmatrix} \varphi_1 \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{n-1} \end{pmatrix},$$

where φ_i corresponds to γ_i. Then we have

$$S = S_1 + \cdots + S_{n-1},$$

where

$$\Omega_i \to S_i u_0 = \begin{pmatrix} 0 & 0 \\ S_{11}^{(i)} & S_{12}^{(i)} \\ S_{21}^{(i)} & S_{22}^{(i)} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \varphi_l \\ \varphi_m \\ 0 \end{pmatrix}.$$

Now, we look for Σ_l and Σ_m such that

$$\begin{pmatrix} \Sigma_l & 0 \\ 0 & \Sigma_m \end{pmatrix} \approx \tilde{S}_i = \begin{pmatrix} S_{11}^{(i)} & S_{12}^{(i)} \\ S_{21}^{(i)} & S_{22}^{(i)} \end{pmatrix},$$

i.e. $(\Sigma_l \varphi_l, \varphi_l) \approx \|\varphi_l\|^2_{H_{00}^{1/2}(\gamma_l)}$ and $(\Sigma_m \varphi_m, \varphi_m) \approx \|\varphi_m\|^2_{H_{00}^{1/2}(\gamma_m)}$. Figure 4.1 gives a typical situation of a subdomain Ω_i with interfaces γ_l and γ_m.

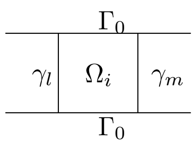

Figure 4.1 Subdomain Ω_i with interfaces γ_l and γ_m.

The corresponding Schur complement is equivalent to the following norm

$$\begin{aligned} \left(S_i \begin{pmatrix} \varphi_l \\ \varphi_m \end{pmatrix}, \begin{pmatrix} \varphi_l \\ \varphi_m \end{pmatrix} \right) &= \inf_{w^h \in H_h(\Omega_i), w^h|_{\gamma_l} = \varphi_l, w^h|_{\gamma_m} = \varphi_m, w^h|_{\Gamma \cap \partial\Omega_i} = 0} |w^h|^2_{H^1(\Omega_i)} \\ &\approx \|\varphi_l^h\|^2_{H_{00}^{1/2}(\gamma_l)} + \|\varphi_m^h\|^2_{H_{00}^{1/2}(\gamma_m)} \end{aligned}$$

due to the previous analysis. Hence, we have the preconditioner for the global Schur complement as a block diagonal matrix.

Let us now consider some γ_ℓ and let us omit the subindex ℓ. A norm which is equivalent to Schur complement norm has the following additive form.

$$\begin{aligned}
&\|\varphi^h\|^2_{H^{1/2}_{00}(\gamma)} \\
&= \sum_{z_i\in\gamma}(\varphi^h(z_i))^2\cdot h + \sum_{z_i, i\neq j}\sum_{z_j}\frac{(\varphi^h(z_i)-\varphi^h(z_j))^2}{|z_i-z_j|^2}h_ih_j + \sum_{z_i\in\gamma}\frac{(\varphi^2(z_i))^2}{(z_i-a)(z_j-b)}h_i \\
&\approx \sum_{z_i\in\gamma}(\tilde\varphi^h(z_i))^2\cdot h + \sum_{z_i, i\neq j}\sum_{z_j}\frac{(\tilde\varphi^h(z_i)-\tilde\varphi^h(z_j))^2}{|z_i-z_j|^2}h_ih_j + \sum_{z_i\in\gamma}\frac{(\tilde\varphi^2(z_i))^2}{(z_i-a)(z_j-b)}h_i \\
&= \|\tilde\varphi^h\|^2_{H^{1/2}_{00}(\tilde\gamma)},
\end{aligned}$$

where, in the second equation, we have everything replaced by its “tilde” (map it onto $[0,1]$) which is for a curved boundary.

Hence consider the square domain. For example, consider the domain with 4 subdomains (Ω_i, $i=1,2,3,4$) whose interfaces (γ_i, $i=1,2,3$) do not meet each other. Then, we have

$$S = \begin{bmatrix} S_1 + S_2^{(1,1)} & S_2^{(1,2)} & \\ S_2^{(2,1)} & S_2^{(2,2)} + S_3^{(1,1)} & S_3^{(1,2)} \\ & S_3^{(2,1)} & S_3^{(2,2)} + S_4 \end{bmatrix}$$

where the submatrix S_i is the Schur complement (S-C) matrix corresponding to the subdomain Ω_i. For instance,

$$S_2 = \begin{bmatrix} S_2^{(1,1)} & S_2^{(1,2)} \\ S_2^{(2,1)} & S_2^{(2,2)} \end{bmatrix}$$

and $S_2^{(i,j)}$ is the S-C matrix corresponding to Ω_2 and γ_i and γ_j. Here, we may write

$$S = \tilde S_1 + \tilde S_2 + \tilde S_3 + \tilde S_4$$

where $\tilde S_i$ is just the extension of S_i by zero elements. Now, in terms of the spectral equivalence, we have

$$S_1 \approx \Sigma_1, \qquad S_2 \approx \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}, \qquad S_3 \approx \begin{bmatrix} \Sigma_2 & \\ & \Sigma_3 \end{bmatrix} \quad \text{and} \quad S_4 \approx \Sigma_3.$$

Note that $(\Sigma_i\phi_i, \phi_i) \approx \|\phi\|^2_{H^{1/2}_{00}(\gamma_i)}$. Hence, we have

$$S \approx \begin{bmatrix} \Sigma_1 & & \\ & \Sigma_2 & \\ & & \Sigma_3 \end{bmatrix}.$$

Given a vertical interface-line segment, we introduce an artificial uniform domain and consider the problem with zero boundary condition on three side except Γ_1 on the left and consider the Schur complement of this problem, denote it by S.

For a given γ_i interface, we suppose that we have the mapping from γ_i onto one side of the rectangular domain with uniform mesh of size $h = 1/n$. Thus, we can now consider our interface problem arising from a rectangular model. The Schur complement of this model problem satisfies the spectral equivalence relation

$$(S\phi, \phi) \approx \|\phi^h\|^2_{H^{1/2}_{00}(\Gamma_1)}.$$

Note 10: Is A_Ω correct?

In the rectangular domain, we have

$$A_\Omega = \begin{bmatrix} A_0 + 2I & -I & & & \\ -I & A_0 + 2I & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & A_0 + 2I & -I \\ & & & -I & \frac{1}{2}A_0 + I \end{bmatrix} := \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix},$$

where

$$A_0 = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix},$$

Note 11: We have ^{t} and $^\top$. Unify?

and

$$\bar{A}_{12} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -I \end{bmatrix}^t = (\bar{A}_{21})^t, \qquad \bar{A}_{22} = \frac{1}{2}A_0 + I.$$

Now we have

$$S = \bar{A}_{22} - \bar{A}_{21}(\bar{A}_{11})^{-1}\bar{A}_{12}$$

and

$$(S\phi, \phi) = \inf_{u^h|_{\Gamma_1} = \phi^h, u^h|_{\partial\Omega\setminus\Gamma_1} = 0} \|u^h\|^2_{H^1(\Omega)} = \inf_{u^h|_{\Gamma_1} = \phi^h, u^h|_{\partial\Omega\setminus\Gamma_1} = 0} (A_\Omega u, u).$$

By the diagonalization, we decompose A_0 as

$$A_0 = Q\Lambda Q^t,$$

where $Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_{n-1} \end{bmatrix}$, Λ is the diagonal matrix with the diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$, and $A_0 q_i = \lambda_i q_i$. Note that it is well known that the eigenvalue $\lambda_i = 4\sin^2 \frac{\pi}{2n}$, the jth component of the eigenvector q_i is $\sqrt{\frac{2}{n}}\sin(\frac{i\pi}{n}j)$, and $QQ^t = I$.

Using this, we get

$$
\begin{aligned}
(\bar{A}_{11})^{-1} &= \begin{bmatrix} Q & & & \\ & Q & & \\ & & \ddots & \\ & & & Q \end{bmatrix} \\
&\times \begin{bmatrix} \Lambda + 2I & -I & & \\ -I & \Lambda + 2I & -I & \\ & \ddots & \ddots & \ddots \\ & & -I & \Lambda + 2I \end{bmatrix}^{-1} \times \begin{bmatrix} Q^t & & & \\ & Q^t & & \\ & & \ddots & \\ & & & Q^t \end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
&\bar{A}_{21}(\bar{A}_{11})^{-1}\bar{A}_{12} \\
&= \begin{bmatrix} 0 & 0 & \cdots & -Q \end{bmatrix} \begin{bmatrix} \Lambda + 2I & -I & & \\ -I & \Lambda + 2I & -I & \\ & \ddots & \ddots & \ddots \\ & & -I & \Lambda + 2I \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ -Q^t \end{bmatrix} \\
&= QB_{22}Q^t,
\end{aligned}
$$

where

$$
B := \begin{bmatrix} \Lambda + 2I & -I & & \\ -I & \Lambda + 2I & -I & \\ & \ddots & \ddots & \ddots \\ & & -I & \Lambda + 2I \end{bmatrix}^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.
$$

Note 12:
I think all matrices above (position of the dots) should look like the left matrix.

Now let us compute the matrix B_{22}. Let $e_i = \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 \end{bmatrix}^\top$, where 1 is in the i-th position. Consider the following matrix equation.

$$
\begin{bmatrix} \Lambda + 2I & -I & & \\ -I & \Lambda + 2I & -I & \\ & \ddots & \ddots & \ddots \\ & & -I & \Lambda + 2I \end{bmatrix} \begin{bmatrix} x_1^{(i)} \\ x_2^{(i)} \\ \vdots \\ x_{n-1}^{(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ e_i \end{bmatrix}.
$$

Then the $(n-1)$-th solution vector of the above matrix equation is the i-th column of the matrix B_{22}, that is,

$$
B_{22} = \begin{bmatrix} x_{n-1}^{(1)} & x_{n-1}^{(2)} & \cdots & x_{n-1}^{(n-1)} \end{bmatrix}.
$$

We denote the vector $x_k^{(i)}$ by

$$x_k^{(i)} = \begin{bmatrix} x_k^{(i)}(1) \\ x_k^{(i)}(2) \\ \vdots \\ x_k^{(i)}(n-1) \end{bmatrix}.$$

Let us consider the j-th component. Then we obtain the following matrix equation

$$\begin{bmatrix} \lambda_j+2 & -1 & & \\ -1 & \lambda_j+2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & \lambda_j+2 \end{bmatrix} \begin{bmatrix} x_1^{(i)}(j) \\ x_2^{(i)}(j) \\ \vdots \\ x_{n-1}^{(i)}(j) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \delta_{ij} \end{bmatrix}.$$

Note 13: Or: details
Note 14: (a) Aren't the + redundant? (b) $x_2^{(i)}$ in line 2? (c) dots at the wrong position?

To discuss some detail we consider the vector equation for a fixed i, i.e.

$$\begin{array}{ccccccc} (\Lambda+2I)x_1^{(i)} & -x_2^{(i)} & & & & = & 0 \\ -x_1^{(i)} & +(\Lambda+2I)x_1^{(i)} & -x_3^{(i)} & & & = & 0 \\ & & \ddots & & & & \\ & -x_{n-3}^{(i)} & & +(\Lambda+2I)x_{n-2}^{(i)} & -x_{n-1}^{(i)} & = & 0 \\ & & & -x_{n-2}^{(i)} & +(\Lambda+2I)x_{n-1}^{(i)} & = & e_i \end{array}$$

The first block corresponds to

$$\begin{array}{cccc} (\lambda+2)x_1^{(i)}(1) & -x_2^{(i)}(1) & = & 0 \\ +(\lambda+2)x_1^{(i)}(2) & -x_2^{(i)}(2) & = & 0 \\ & & \vdots & \\ +(\lambda+2)x_1^{(i)}(n-1) & -x_2^{(i)}(n-1) & = & 0 \end{array}$$

We collect j-th line. If $i=j$, we have

$$x_{n-1}^{(i)} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ x_{n-1}^{(i)}(i) \\ \vdots \\ 0 \end{bmatrix}.$$

Combining the vectors $x_{n-1}^{(i)}$, we can obtain the matrix B_{22} as

$$B_{22} = \begin{bmatrix} x_{n-1}^{(1)}(1) & 0 & \cdots & 0 \\ 0 & x_{n-1}^{(2)}(2) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x_{n-1}^{(n-1)}(n-1) \\ . & & & \end{bmatrix}.$$

Then

Note 15: Shall we delte the last row of B_{22}?

$$\tilde{S} = \bar{A}_{22} - \bar{A}_{21}\bar{A}_{11}^{-1}\bar{A}_{12} = \frac{1}{2}A + I - QB_{22}Q^{\top} = Q(\frac{1}{2}\Lambda + I - B_{22})Q^{\top}$$

and the i-th eigenvalue of S is

$$\lambda_i(\tilde{S}) = \frac{1}{2}\lambda_i + 1 - x_{n-1}^{(i)}(i).$$

To compute $x_{n-1}^{(i)}(i)$, we have to solve the following system of algebraic equations:

$$\begin{bmatrix} \lambda_i + 2 & -1 & & \\ -1 & \lambda_i + 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & \lambda_i + 2 \end{bmatrix} \begin{bmatrix} x_1^{(i)}(i) \\ x_2^{(i)}(i) \\ \vdots \\ x_{n-1}^{(i)}(i) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Let $\alpha_i = \frac{1}{2}\lambda_i + 1$. By using Gauss-elimination technique (multiplying the j-th row by $2\alpha_i$ and adding the $j-1$-th row to j-th row) we obtain the triangular system

$$\begin{bmatrix} d_1 & -d_0 & & 0 \\ & d_2 & -d_1 & \\ & & \ddots & -d_{n-3} \\ 0 & & & d_{n-1} \end{bmatrix} \begin{bmatrix} x_1^{(i)}(i) \\ x_2^{(i)}(i) \\ \vdots \\ x_{n-1}^{(i)}(i) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ d_{n-2} \end{bmatrix},$$

where $d_0 = 1$, $d_1 = 2\alpha_i$, and $d_{j+1} = 2\alpha_i d_j - d_{j-1}$, for $j = 1, 2, \ldots, n-2$. Let $U_n(x)$ be the second kind Chebyshev polynomial of degree n, that is,

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\Big((x + \sqrt{x^2-1})^{n+1} - (x + \sqrt{x^2-1})^{-(n+1)}\Big).$$

Then $d_j = U_j(\alpha_i)$ and

Note 16: Re-sorted paragraph. Correct?

$$x_{n-1}^{(i)}(i) = \frac{d_{n-2}}{d_{n-1}} = \frac{U_{n-2}(\alpha_i)}{U_{n-1}(\alpha_i)}.$$

(We note that the first kind Chebyshev polynomials of degree n at α_j are determined by the same recursion scheme for d_j, but with the initial condition $d_0 = 1$ and $d_1 = \alpha_i$.)

Hence,

$$\begin{aligned}
\lambda_i(\tilde{S}) &= \alpha_i - \frac{d_{n-2}}{d_{n-1}} \\
&= \alpha_i - \frac{U_{n-2}(\alpha_i)}{U_{n-1}(\alpha_i)} \\
&= \alpha_i - \frac{(\alpha_i + \sqrt{\alpha_i^2 - 1})^{n-1} - (\alpha_i + \sqrt{\alpha_i^2 - 1})^{-n+1}}{(\alpha_i + \sqrt{\alpha_i^2 - 1})^{n} - (\alpha_i + \sqrt{\alpha_i^2 - 1})^{-n}} \\
&= \sqrt{\alpha_i^2 - 1}\, \frac{(\alpha_i + \sqrt{\alpha_i^2 - 1})^{n} + (\alpha_i + \sqrt{\alpha_i^2 - 1})^{-n}}{(\alpha_i + \sqrt{\alpha_i^2 - 1})^{n} - (\alpha_i + \sqrt{\alpha_i^2 - 1})^{-n}} \\
&= \sqrt{\alpha_i^2 - 1} f(x),
\end{aligned}$$

where

$$f(x) = \frac{x + 1/x}{x - 1/x}, \quad x = \left(\alpha_i + \sqrt{\alpha_i^2 - 1}\right)^n .$$

Using $\sqrt{\alpha_i^2 - 1} = \sqrt{\lambda_i}\sqrt{1 + \frac{\lambda_i}{4}}$, we have the following estimates for $\lambda_i(\tilde{S})$:

$$\sqrt{\lambda_i} \le \lambda_i(\tilde{S}) \le \sqrt{\lambda_i}\, C(\lambda_{\min}, \lambda_{\max}),$$

where

$$C(\lambda_{\min}, \lambda_{\max}) = \sqrt{1 + \frac{\lambda_{\max}}{4}} \cdot \frac{\beta^n + \beta^{-n}}{\beta^n - \beta^{-n}},$$

with

$$\beta = 1 + \frac{1}{2}\lambda_{\min} + \sqrt{\lambda_{\min} + \frac{1}{4}\lambda_{\min}^2}.$$

Since $\lambda_{\min} = 4\sin^2(\frac{\pi}{2n}) \simeq \frac{1}{n^2}$ and $\lambda_{\max} \le 4$, we have

$$\beta^n \ge (1 + \sqrt{\lambda_{\min}})^n \simeq O(1).$$

Hence, by setting $\Sigma := A^{\frac{1}{2}} = Q\Lambda^{\frac{1}{2}}Q^\top$, we arrive at the following inequality:

$$(\Sigma\phi, \phi) \le (\tilde{S}\phi, \phi) \le C(\Sigma\phi, \phi).$$

Note 17: There are both notations $A^{1/2}$ and $A^{\frac{1}{2}}$. Unify?

Thus, we have

$$\tilde{S} = QJQ^\top \approx A^{1/2}, \text{ with} J = \text{diag}(\lambda_i(\tilde{S})),$$

and

$$\Sigma^{-1} = Q\Lambda^{-\frac{1}{2}}Q^\top,$$

where $Q = (q_1, \ldots, q_{n-1}), q_i(j) = \sqrt{\frac{2}{n}}\sin\frac{i\pi j}{n}$. Since $\lambda_{\max} \le 4$, we have $C \le \frac{5\sqrt{2}}{3}$. If we use the Fast Fourier transform (FFT) algorithm, then the cost for computing $\Sigma^{-1}\phi$ is of order $h^{-1}\log(h^{-1})$. What is $\|\phi^h\|^2_{H^{1/2}}$? Since (as a discrete inner product on Γ)

$$(\phi^h, \psi^h)_{L_{2,h}} = h(\phi, \psi),$$

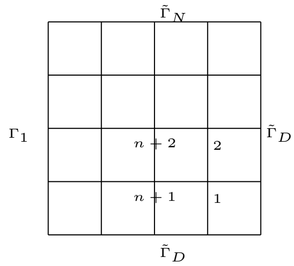

Figure 4.2 Domain and Grid Numbering.

where the right hand side is vector inner product and

$$\|\phi^h\|_{H^1}^2 = (\frac{1}{h^2}A\phi,\phi)_{L_{2,h}} = \frac{1}{h}(A\phi,\phi),$$

we have

$$\|\phi^h\|_{H^{1/2}}^2 \simeq \left(\left(\frac{1}{h^2}A\right)^{1/2}\phi,\phi\right)_{L_{2,h}} = (A^{\frac{1}{2}}\phi,\phi).$$

Thus, we are done with the case of Dirichlet boundary conditions. Next we consider mixed boundary conditions. Recall that, for Dirichlet boundary conditions, we have

$$(S\varphi,\varphi) \approx \sum_l (\Sigma_l\varphi,\varphi), \qquad \Sigma_l = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 2 \end{bmatrix}^{1/2}.$$

For mixed boundary conditions, we have

$$(S\varphi,\varphi) \approx \sum_l \|\varphi^h\|_{\breve{H}^{1/2}(\gamma_l)}^2,$$

where a_l is the endpoint of the interface γ_l lying on the Dirichlet boundary, and

$$\|\varphi\|_{\breve{H}^{1/2}(\gamma_l)}^2 = \|\varphi\|_{H^{1/2}(\gamma_l)}^2 + \int_{\gamma_l} \frac{\varphi^2(x)}{|x-a_l|}\,dx.$$

Let

$$A_1 = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & & & \\ & \ddots & 0 & \\ & & \ddots & \\ & 0 & 1 & \\ & & & 1/2 \end{bmatrix},$$

and

$$A_\Omega = \begin{bmatrix} A_1+2D & -D & & & \\ -D & A_1+2D & -D & & \\ & \ddots & \ddots & \ddots & \\ & & -D & A_1+2D & -D \\ & & & -D & \frac{1}{2}A_1+D \end{bmatrix}.$$

We note that A_1 corresponds to the first right vertical block. Now, we obtain

$$\begin{aligned}(A_\Omega u, u) &= \sum_{x(i,j)\in\Omega} \{(u_{i,j}-u_{i-1,j})^2 + (u_{i,j}-u_{i,j-1})^2\} \\ &\quad + \frac{1}{2}\sum_{j=1}^{n}(u_{n,j}-u_{n,j-1})^2 \; + \; \frac{1}{2}\sum_{i=1}^{n}(u_{i,n}-u_{i-1,n})^2 \\ &\approx (B_\Omega u, u),\end{aligned}$$

where

$$B_\Omega = \begin{bmatrix} A_1+2I & -I & & & \\ -I & A_1+2I & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & A_1+2I & -I \\ & & & -I & \frac{1}{2}A_1+I \end{bmatrix}.$$

The following lemma holds:

Lemma 4.1 *If $A \sim B$ then the corresponding Schur complements are also spectrally equivalent, i.e. $S_A \sim S_B$.*

Furthermore, we have $\lambda_{\min}(A_1) = O(h^2)$, $\lambda_{\max}(A_1) = O(1)$, and

$$S \approx \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}^{1/2}.$$

In this case the eigenvectors cannot be easily constructed. Thus, we consider the representation

$$
\begin{aligned}
S &= \left(\frac{1}{2}A_1 + D\right) - \begin{bmatrix} 0 & \cdots & 0 & -D \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \cdot & \\ \ddots & \ddots & \ddots & \\ & -D & A_1 + 2D & -D \\ & \ddots & \ddots & \ddots \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -D \end{bmatrix} \\
&= D\left(\frac{1}{2}D^{-1}A_1 + I\right) \\
&\quad -D\begin{bmatrix} 0 & \cdots & 0 & -I \end{bmatrix} \left(\begin{bmatrix} D & & \\ & \ddots & \\ & & D \end{bmatrix} \begin{bmatrix} D^{-1}(A_1+2I) & -I & \\ -I & \ddots & \ddots \\ & \ddots & \ddots \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -D \end{bmatrix} \\
&= D\left(\frac{1}{2}D^{-1}A_1 + I\right) \\
&\quad -D\begin{bmatrix} 0 & \cdots & 0 & -I \end{bmatrix} \begin{bmatrix} D^{-1}(A_1+2I) & -I & \\ -I & \ddots & \ddots \\ & \ddots & \ddots \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -I \end{bmatrix}.
\end{aligned}
$$

The following matrix corresponds to the finite difference version for the one-side Neumann problem, see [37, 39]:

Note 18: First row of above matrix correct?

$$
A_2 = D^{-1}A_1 = \begin{bmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{bmatrix}.
$$

We obtain

$$
A_2 = Q_2\Lambda_2 Q_2^{-1} = Q_2\Lambda_2 Q_2^\top D,
$$

where

Note 19: Is Q_2 a matrix, i.e. $[q_1 \ \cdots \ q_n]$?

$$
Q_2 = [q_1,\ q_2,\ \ldots], \qquad q_i(j) = \sqrt{\frac{2}{n}}\sin\frac{(2i-1)\pi j}{2n}, \qquad \lambda_i = 4\sin^2\frac{(2i-1)\pi}{2n},
$$

for $i, j = 1, \ldots, n$. Here $\tilde{\Lambda}_2$ is obtained from the Chebysheff polynomial. Therefore,

Note 20: We have Chebysheff and Chebyshev. Which notation shall we use?

$$
S = DQ_2\tilde{\Lambda}_2 Q_2^\top D \approx DQ_2\Lambda_2^{1/2}Q_2^\top D = \Sigma_{DN}, \qquad \Sigma_{DN}^{-1} = Q_2\Lambda_2^{-1/2}Q_2^\top.
$$

For the implementation, it is possible to use FFT for Q_2. In fact, we have a D-orthogonal basis such that the following properties hold:

- $A_2 q = D^{-1} A_1 q = \lambda q \implies A_1 q = \lambda D q,$
- $(Dq_i, q_j) = \delta_{ij},$
- $(D^{-1/2} A_1 D^{-1/2}) D^{1/2} q = \lambda D^{1/2} q,$
- $(Dq_i, q_j) = (\tilde{q}_i, \tilde{q}_j) = \delta_{ij} \implies Q_2^\top D Q = I \implies Q_2^{-1} = Q^\top D,$ with $\tilde{q} = D^{1/2} q.$

Neumann Boundary Conditions both on top and bottom of boundary

In this case, we have

$$A_3 = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}, \qquad Q_3 = \begin{bmatrix} 1/2 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1/2 \end{bmatrix}$$

Repeating the same analysis, we arrive at the following two possibilities (see Figure 4.3):

$$\Sigma_{NN} = A_3^{1/2} + \frac{1}{n} I, \qquad \Sigma_{NN}^{(N)} = A_3^{1/2}$$

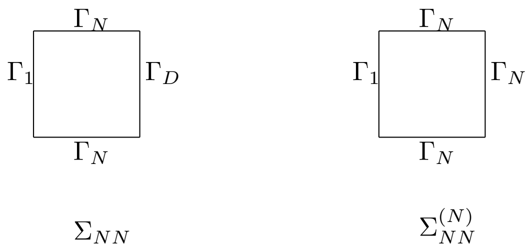

Figure 4.3 The two possibilities.

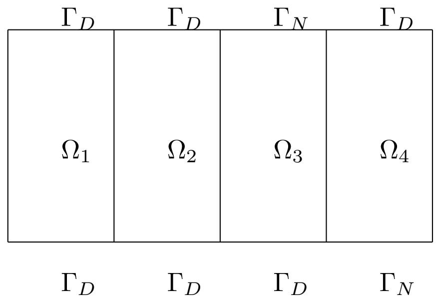

Figure 4.4 Domain partition with boundary conditions.

This problem generates the following Schur complements and their preconditioners:

$$\begin{aligned}
\Omega_1 \quad &\longrightarrow \quad S^{(1)} \approx \Sigma^{(1)}_{DD}, \\
\Omega_2 \quad &\longrightarrow \quad S^{(2)} = \begin{bmatrix} S^{(2)}_{11} & S^{(2)}_{12} \\ S^{(2)}_{21} & S^{(2)}_{22} \end{bmatrix} \approx \begin{bmatrix} \Sigma^{(1)}_{DD} & \\ & \Sigma^{(2)}_{DD} \end{bmatrix}, \\
\Omega_3 \quad &\longrightarrow \quad S^{(3)} = \begin{bmatrix} S^{(3)}_{11} & S^{(3)}_{12} \\ S^{(3)}_{21} & S^{(3)}_{22} \end{bmatrix} \approx \begin{bmatrix} \Sigma^{(1)}_{DN} & \\ & \Sigma^{(3)}_{DN} \end{bmatrix}, \\
\Omega_4 \quad &\longrightarrow \quad S^{(4)} \approx \Sigma^{(3)}_{DN},
\end{aligned}$$

and, therefore,

$$\Sigma = \begin{bmatrix} \Sigma^{(1)}_{DD} & & \\ & \Sigma^{(2)}_{DD} + \Sigma^{(2)}_{DN} & \\ & & \Sigma^{(3)}_{DN} + \Sigma^{(3)}_{ND} \end{bmatrix} \approx \begin{bmatrix} \Sigma^{(1)}_{DD} & & \\ & \Sigma^{(2)}_{DD} & \\ & & \Sigma^{(3)}_{DD} \end{bmatrix}.$$

Here $\Sigma^{(2)}_{DN}$ is smaller than $\Sigma^{(2)}_{DD}$ and $\Sigma^{(3)}_{DN} + \Sigma^{(3)}_{ND} \approx \Sigma^{(3)}_{DD}$, i.e.

$$\begin{aligned}
\Sigma^{(2)}_{DD} \leq \Sigma^{(2)}_{DD} + \Sigma^{(2)}_{DN} \leq c\Sigma_{DD} \qquad &\text{on } \gamma_2, \\
\Sigma^{(3)}_{DN} + \Sigma^{(3)}_{ND} \approx \Sigma^{(3)}_{DD} \qquad &\text{on } \gamma_3.
\end{aligned}$$

We note that $(\Sigma^{(3)}_{DN})^{-1} + (\Sigma^{(3)}_{ND})^{-1} \neq (\Sigma_{DN} + \Sigma_{ND})^{-1}$. Furthermore, we have

$$(\Sigma_{DN}\varphi, \varphi) \approx \|\varphi^h\|^2_{H^{1/2}} + \int \frac{(\varphi^h(x))^2}{x - a_3}\, dx$$

and

$$\Sigma = \begin{bmatrix} \Sigma^{(1)}_{DD} & & \\ & \Sigma^{(2)}_{NN} & \\ & & \Sigma^{(3)}_{DD} \end{bmatrix} \not\approx S.$$

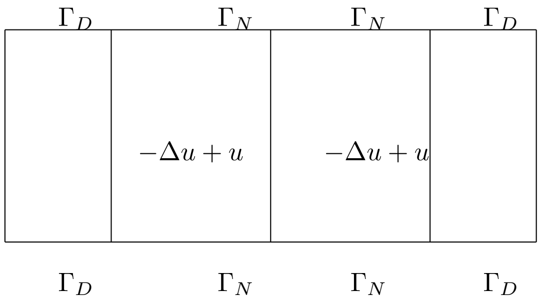

Figure 4.5 Other boundary conditions.

This is only related to the semi-norm. What should we do in this case? We use $-\Delta + I$ instead of $-\Delta$ and construct a preconditioner for $-\Delta u + u$. Hence, we have $\Sigma_{NN}^{(2)}$ in the second block of above expression.

Lemma 4.2 *Let*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

Then $B_{11}^{-1} = A_{11} - A_{12}A_{22}^{-1}A_{21}$.

Proof. From the obvious relation

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix},$$

we immediately obtain the equations

$$B_{11}A_{11} + B_{12}A_{21} = I_1, \quad B_{11}A_{12} + B_{12}A_{22} = 0,$$
$$B_{21}A_{11} + B_{22}A_{21} = 0, \quad B_{21}A_{12} + B_{22}A_{22} = I_2,$$

which lead to the relations

$$\Longrightarrow \quad B_{12} = -B_{11}A_{12}A_{22}^{-1}, \quad B_{11}A_{11} - B_{11}A_{12}A_{22}^{-1}A_{21} = I_1,$$
$$\Longrightarrow \quad B_{11}^{-1} = A_{11} - A_{12}A_{22}^{-1}A_{21}.$$

This completes the proof of the lemma. □

We note that a similar result holds for B_{22}.

The cross-point case is more complicated. Let us consider some model boundary value problem in the domain Ω which consists of four subdomains as is shown in

Figure 4.6. Let us construct a preconditioner $\Sigma^{(i)}$ for $S^{(i)}$ on each subdomain. Then we

Note 21: Then we ...?

$$S = S^{(1)} + S^{(2)} + S^{(3)} + S^{(4)} \approx \Sigma^{(1)} + \Sigma^{(2)} + \Sigma^{(3)} + \Sigma^{(4)} = \Sigma.$$

However, it is not immediately clear how one can efficiently solve the preconditioning system. The trace theorem is not enough. We need the so-called "Schwarz machinery", in particular, the theory of the Additive Schwarz Methods (ASM) will be very helpful.

2	1
3	4

Figure 4.6 The cross point case.

5 The Schwarz Alternating Method

In 1869, H. A. Schwarz introduced an overlapping domain decomposition method [40]. He used this method, which is now called the Schwarz Alternating Method, for proving the existence of harmonic functions in complicated domains composed of simpler domains where the existence is known.

Let us consider again the Dirichlet problem for the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{5.1}$$

in some domain $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ which is composed of two overlapping subdomains Ω_1 and Ω_2. Then the Schwarz Alternating Method is nothing but an iterative process, where we alternately solve Dirichlet problems in the two subdomains, i.e. starting with some initial guess u^0 that vanishes on Γ, we perform following iteration steps:

Note 22: Or: we perform following iteration steps for $k = 1, 2, \ldots$:

1. Solution on Ω_1: Determine u_{2k+1} such that

$$\begin{cases} -\Delta u_{2k+1} = f - \Delta u^{2k}, \\ u_{2k+1} = 0 \quad \text{in } \Gamma_1, \end{cases}$$

   and set $u^{2k+1} = u^{2k} + u_{2k+1}$.

2. Solution on Ω_2: Determine u_{2k+2} such that

$$\begin{cases} -\Delta u_{2k+2} = f - \Delta u^{2k+1}, \\ u_{2k+2} = 0 \quad \text{in } \Gamma_1, \end{cases}$$

   and set $u^{2k+2} = u^{2k+1} + u_{2k+2}$, where $k = 1, 2, \ldots$

Let us recall the weak formulation of the boundary value problem (5.1): Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = l(v) \quad \forall v \in H_0^1(\Omega),$$

where

$$a(u, v) = \int_\Omega (\nabla u, \nabla v)\, d\Omega \quad \text{and} \quad l(v) = \int f v\, d\Omega.$$

The weak formulation of the Schwarz alternating method, which was first given by S. L. Sobolev in [42], reads as follows:

$$u_{2k+1} \in H_0^1(\Omega_1):$$

$$a(u^{2k} + u_{2k+1}, v) = l(v) \quad \forall v \in H_0^1(\Omega_1)$$

$$u^{2k+1} = u^{2k} + u_{2k+1}$$

$$u_{2k+2} \in H_0^1(\Omega_2):$$

$$a(u^{2k+1} + u_{2k+2}, v) = l(v) \quad \forall v \in H_0^1(\Omega_2)$$

$$u^{2k+2} = u^{2k+1} + u_{2k+2}.$$

Let us now introduce the abbreviations

$$H = H_0^1(\Omega), \quad H_1 = H_0^1(\Omega_1), \quad H_2 = H_0^1(\Omega_2)$$

and the orthogonal projection (the Ritz projection) $P_i : H \to H_i$ with respect to the bilinear form $a(u, v)$. Then we can do the following analysis:

Note 23: Maybe some more words between the formulae?

$$a(u_{2k+1}, v) = l(v) - a(u^{2k}, v) = a(u, v) - a(u^{2k}, v) = a(u - u^{2k}, v) \quad \forall v \in H_1$$

$$\begin{cases} u_{2k+1} = P_1(u - u^{2k}) \\ u^{2k+1} = u^{2k} + P_1(u - u^{2k}) \\ u^{2k+2} = u^{2k+1} + P_2(u - u^{2k+1}) \end{cases} \quad \Longleftarrow \quad \begin{cases} u^{2k+1} - u \\ = u^{2k} - u + P_1(u - u^{2k}) \\ = (I - P_1)(u - u^{2k}) \end{cases}$$

$$\psi^k = u^k - u$$

$$Q_i : H \to H_1^\perp, \quad Q_i = I - P_i$$

$$\psi^{2k+1} = (I - P_1)\psi^{2k} = Q_1 \psi^{2k}$$

$$\psi^{2k+2} = (I - P_2)\psi^{2k+1} = Q_2 \psi^{2k+1}$$

$$
\begin{aligned}
& k \geq 1 \\
a(\psi^{2k+1}, \psi^{2k+1}) &= \|\psi^{2k+1}\|_a^2 = \|Q_1 \psi^{2k}\|_a^2 \\
&= \|Q_1 Q_2 \psi^{2k}\|_a^2 = a(Q_1 Q_2 \psi^{2k}, Q_1 Q_2 \psi^{2k}) \\
&= a(Q_2 Q_1 Q_1 Q_2 \psi^{2k}, \psi^{2k}) = a(Q_2 Q_1 Q_2 \psi^{2k}, \psi^{2k}) \\
&= a((I - (P_1 + P_2) + P_1 P_2 + P_2 P_1 - P_2 P_1 P_2)\psi^{2k}, \psi^{2k}) \\
&= a(\psi^{2k}, \psi^{2k}) - a((P_1 + P_2)\psi^{2k}, \psi^{2k}) + a(P_1 P_2 \psi^{2k}, \psi^{2k}) \\
&\quad + a(P_2 P_1 \psi^{2k}, \psi^{2k}) - a(P_2 P_1 P_2 \psi^{2k}, \psi^{2k}) \\
&= a(\psi^{2k}, \psi^{2k}) - a((P_1 + P_2)\psi^{2k}, \psi^{2k})
\end{aligned}
$$

Let us assume that there exists a positive constant $\alpha \leq 1$ such that

$$\alpha a(u, u) \leq a((P_1 + P_2)u, u) \quad \forall u \in H.$$

Then we easily get the estimates

$$
\begin{cases}
\|\psi^{2k+1}\|_a \leq (1-\alpha)^{1/2} \|\psi^{2k}\|_a \\
\|\psi^{2k+2}\|_a \leq (1-\alpha)^{1/2} \|\psi^{2k+1}\|_a
\end{cases}
$$

which finally yield the convergence rate estimate

$$\|\psi^{2k+2}\|_a \leq (1-\alpha) \|\psi^{2k}\|_a$$

for the Schwarz alternating method in the energy norm $\|\cdot\|_a$.

The error propagation operator of the Schwarz alternating method is multiplicative. Therefore, such types of methods are also called multiplicative Schwarz methods. Multiplicative Schwarz methods are not in parallel. To construct a parallel method we will consider an additive version of the Schwarz method in the next section.

6 Additive Schwarz Method

The Additive Schwarz Method (ASM) was suggested by A. Matsokin and S. Nepomnyaschikh in 1985 [21].

In the case of two subdomains, the ASM is based on the following inequalities

$$\alpha\, a(u, u) \leq a((P_1 + P_2)u, u) \leq 2\, a(u, u) \quad \forall u \in H,$$

and can be written in the following form: Given initial guess $u^0 \in H$, find iteratively

$$u^{k+1} = u^k - \tau_k (P_1 + P_2)(u^k - u) \quad k = 0, 1, 2, \ldots$$

The general theory of ASM is given by the following abstract theorem:

Theorem 6.1 *Let H be a Hilbert space equipped with the inner product (u, v). Let us consider the decomposition of H into subspaces H_i, i.e. $H = H_1 + H_2 + \cdots + H_m$. Moreover, let $A : H \to H$ be a symmetric, bounded and positive definite operator on H and denote by $a(u, v) = (Au, v)$ the corresponding bilinear form. Finally, let $P_i : H \to H_i$ be the orthogonal projections with respect to $a(u, v)$. Then, the following two statements are equivalent:*

a) There exists an $\alpha > 0$ such that, for all $u \in H$, there exists a decomposition

$$u = u_1 + u_2 + \cdots + u_m, \quad \textit{with } u_i = H_i,$$

satisfying the inequality

$$\alpha \left(a(u_1, u_1) + a(u_2 + u_2) + \cdots + a(u_m, u_m)\right) \leq a(u, u).$$

b) The inequality

$$\alpha\, a(u, u) \leq a((P_1 + P_2 + \cdots + P_m)u, u)$$

holds for all $u \in H$.

Proof. For the proof of the implication b) $\Longrightarrow$ a), we introduce

$$P = P_1 + P_2 + \cdots + P_m.$$

The operator P is symmetric, bounded and positive definite. For any $u \in H$, there exists a $v \in H$ such that $u = Pv = \sum_{i=1}^m P_i v$. We set $u_i = P_i v$. Then, we get

$$\begin{aligned} \sum_{i=1}^m a(u_i, u_i) &= \sum_{i=1}^m a(P_i v, P_i v) = \sum_{i=1}^m a(P_i v, v) = a(\sum_{i=1}^m P_i v, v) \\ &= a(u, v) = a(u, P^{-1}u) \leq \frac{1}{\alpha} a(u, u). \end{aligned}$$

This proves the first assertion.

In order to prove a) $\Longrightarrow$ b), we start with the equality

$$\|u\|_a = \sup_{v \in H} \frac{a(u, v)}{\|v\|_a}. \tag{6.1}$$

Indeed,

$$\sup_{v \in H} \frac{a(u, v)}{\|v\|_a} \overset{\text{C.B.}}{\leq} \sup_{v \in H} \frac{\|u\|_a \|v\|_a}{\|v\|_a} = \|u\|_a.$$

With $v = u$, we get $\sup \frac{a(u,v)}{\|v\|_a} \geq \frac{a(u,u)}{\|u\|_a} = \|u\|_a$. This proves (6.1).

Now, for any $u \in H$, we can conclude

$$\begin{aligned}
\|u\|_a &= \sup_{v\in H} \frac{a(u,v)}{\|v\|_a} = \sup_{v\in H} \frac{a(u, \sum_{i=1}^m v_i)}{\|v\|_a} \\
&= \sup_{v\in H} \sum_{i=1}^m \frac{a(u, P_i v_i)}{\|v\|_a} = \sup_{v\in H} \sum_{i=1}^m \frac{a(P_i u, v_i)}{\|v\|_a} \\
&\overset{\text{C.B.}}{\le} \sup_{v\in H} \frac{\sum_{i=1}^m \|P_i u\|_a \|v_i\|_a}{\|v\|_a} \\
&\overset{\text{C.B.}}{\le} \sup_{v\in H} \frac{\sqrt{\sum_{i=1}^m \|P_i u\|_a^2} \cdot \sqrt{\sum_{i=1}^m \|v_i\|_a^2}}{\|v\|_a} \\
&\le \frac{1}{\sqrt{\alpha}} \sqrt{\sum_{i=1}^m \|P_i u\|_a^2}
\end{aligned}$$

using (6.1), the Cauchy–Bunjakowski (C.B.) inequality two times and (a). □

Let Ω be a union of two overlapping subdomains Ω_1 and Ω_2. We want to show that, for any $u \in H_0^1(\Omega)$, there exist $u_i \in H_0^1(\Omega)$ such that $u_1 + u_2 = u$ and

$$\|u_1\|^2_{H^1(\Omega_1)} + \|u_2\|^2_{H^1(\Omega_2)} \le \frac{1}{\alpha} \|u\|^2_{H^1(\Omega)}.$$

Let

$$u_1(x) = \begin{cases} u(x), & x \in \Omega_1 \backslash \Omega_2 \\ \text{extension}, & x \in \Omega_1 \cap \Omega_2. \end{cases}$$

The extension implies the estimate $\|u_1\|_{H^1(\Omega_1)} \le C\|u\|_{H^1(\Omega)}$ with $u_1 \in H_0^1(\Omega_1)$. Then, let $u_2 = u - u_1$ with $u_2 \in H_0^1(\Omega_2)$. This gives the estimates

$$\|u_2\|_{H^1(\Omega_2)} \le \|u\|_{H^1(\Omega)} + \|u_1\|_{H^1(\Omega_1)} \le (1 + C)\|u\|_{H^1(\Omega)}$$

where the constant depends on the extension.

In general, we have $a(Pu, u) \le m \cdot a(u, u)$ where m is the number of subspaces. However, if m is large, a refined estimate is required.

Theorem 6.2 *The following two assertions are equivalent:*

a) $a(Pu, u) \le \beta\, a(u, u)\ \forall u \in H$,

b) $a(u, u) \le \beta \inf_{u_1+\cdots+u_m=u, u_i\in H_i} \sum_{i=1}^m a(u_i, u_i)$.

Proof. Let $u \in H$ and put $u_i = P_i P^{-1} u$. Then, we have

$$u_1 + \cdots + u_m = P_1 P^{-1} u + \cdots + P_m P^{-1} u = u.$$

Let $v_i \in H_i : v_1 + \cdots + v_m = u$ be another decomposition with $v_i = u_i + w_i$. Obviously,

Note 24: Or: Let $v_i \in H_i$ with … that satisfies $v_i = u_i + w_i$.

one obtains $\sum_{i=1}^m w_i = 0$. Moreover,

$$\begin{aligned}\sum_{i=1}^m a(v_i, v_i) &= \sum_{i=1}^m a(u_i, u_i) + 2a(u_i, w_i) + a(w_i, w_i) \\ &= \sum_{i=1}^m a(u_i, u_i) + 2a(P_i P^{-1} u, w_i) + a(w_i, w_i) \\ &= \sum_{i=1}^m a(u_i, u_i) + 2a(P^{-1}u, \sum_{i=1}^m w_i) + \sum_{i=1}^m a(w_i, w_i).\end{aligned}$$

Hence,

$$\begin{aligned}\inf_{u=v_1+\cdots+v_m, v_i \in H_i} \sum_{i=1}^m a(v_i, v_i) &= \sum_{i=1}^m a(u_i, u_i) = \sum_{i=1}^m a(P_i P^{-1} u, P_i P^{-1} u) \\ &= \sum_{i=1}^m a(P^{-1}u, P_i P^{-1} u) = a(P^{-1}u, P P^{-1} u) \\ &= a(P^{-1}u, u).\end{aligned}$$

Choosing $u = P^{1/2} v$, we can prove

$$a(Pu, u) \leq \beta\, a(u, u)\ \forall u \in H \iff a(u, u) \leq \beta\, a(P^{-1}u, u)\ \forall u \in H.$$

This proves the theorem. □

Lemma 6.3 *Let* $a(u, v) = A(u, v)$*. Moreover, let us define*

$$A_i : H_i \to H_i, \quad (A_i u_i, v_i) = A(u_i, v_i) \quad \forall u_i, v_i \in H_i.$$

Finally, let Q_i *be the orthogonal projector with respect to* $(\cdot, \cdot)$*, i.e.*

$$Q_i : H \to H_i \quad \textit{in } (\cdot, \cdot).$$

Then, we have $P_i = A_i^{-1} Q_i A$*.*

Proof. Let $u \in H$ and set $u_i = P_i u$, $w_i = A_i^{-1} Q_i u$. This gives $A w_i = Q_i A u$. For all $v_i \in H_i$, we can conclude that

$$a(w_i, v_i) = (A w_i, v_i) = (A_i w_i, v_i) = (Q_i A u, v_i) = (Au, v_i) = a(u, v_i).$$

Hence, we have $w_i = u_i$. □

Let us recall that $a(u, v) = (u, v)_{H^1}$ and $(u, v) = (u, v)_{L_2}$ for our model problem.

Let us summarize the results obtained so far. The previous lemma has given the relation between the projection P_i corresponding to the given bilinear form $a(\cdot, \cdot)$ and the L_2 projection Q_i. That is, we began with

- the decomposition $H = H_1 + H_2 + \cdots + H_m$,
- the bilinear form $a(u, v) = (Au, v)$, and
- the energetic projections $P_i : H \to H_i$.

For $P = \sum_{i=1}^m P_i$, we have shown the inequalities

$$\alpha\, a(u, u) \le a(Pu, u) \le \beta\, a(u, u) \quad \forall u \in H.$$

Then, we have proved that

$$P_i = A_i^{-1} Q_i A$$

where $Q_i : H \to H_i$ are the projections in $(\cdot, \cdot)$. Now, we have

$$\alpha\, (Au, u) \le (A(\sum_{i=1}^{m} A_i^{-1} Q_i A)u, u) \le \beta\, (Au, u) \quad \forall u \in H$$

and this is equivalent to

$$\alpha\, (Au, u) \le (AB^{-1}Au, u) \le \beta\, (Au, u) \quad \forall u \in H,$$

where $B^{-1} = \sum_{i=1}^m Q_i A_i^{-1} Q_i$. Putting $Au = v$, one easily concludes that

$$\alpha\, (A^{-1}v, v) \le (B^{-1}v, v) \le \beta\, (A^{-1}v, v) \quad \forall v \in H,$$

or

$$\alpha\, (Bu, u) \le (Au, u) \le \beta\, (Bu, u) \quad \forall v \in H.$$

Thus, we have constructed a preconditioner B which is equivalent to A. Now, we can use B as a preconditioner in the Richardson iteration

$$u^{k+1} = u^k - \tau_k B^{-1}(Au^k - f),$$

or in conjugate gradient method for solving our original problem $Au = f$.

Example 6.4 (Simple one-dimensional example) We consider the equation $-u'' = f$ in $\Omega = (0, 1)$ with the homogeneous Dirichlet boundary condition $u(0) = u(1) = 0$. Then we have $Au = f$ with

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

Note that A is a $(n-1) \times (n-1)$ matrix and $H = R^{n-1}$. As in Figure 6.1, we define $H = H_1 + H_2$, $\Omega = \Omega_1 \cup \Omega_2$ and

$$u = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{bmatrix}^\top$$

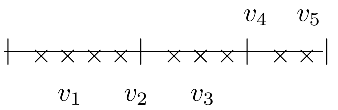

Figure 6.1 Numbering of (i,j) element and its subdivision.

where $H_1 = \{(v_1, v_2, v_3, 0, 0)^t\}$ and $H_2 = \{(0, 0, v_3, v_4, v_5)^t\}$. Here the vectors v_1, v_3 and v_5 correspond to the interior nodes inside the intervals, whereas the values u_2 and u_4 correspond to the interface nodes between the intervals. Now we can take $\beta = 2$ by the property of the projection. For $u \in H$, we want to find some α such that

$$\alpha \sum_{i=1}^{2} a(u_i, u_i) \leq a(u, u), \quad u = u_1 + u_2,$$

where $u_i \in H_i$. From Figure 6.1, it can be observed

$$(Au_1, u_1) \leq c\,(Au, u),$$

where c is independent of h. Setting $u_2 = u - u_1$, we have such an α which is independent of h. In this case, we have

$$Q_1 = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix}.$$

Moreover,

$$Q_1 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \\ 0 \end{bmatrix}, \quad Q_2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}$$

and

$$A_1 = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & 0 \\ & & & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & & & \\ 0 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

Note 25: "multiplication with v_4, v_5-multiplication" correct?

Note that A_1 has zero entries that correspond to the multiplication with v_4, v_5-multiplication, whereas A_2 has zero entries corresponding to the v_1, v_2-multiplication. Now, we obtain the preconditioner in the form

$$B^{-1} = Q_1 A_1^+ Q_1 + Q_2 A_2^+ Q_2 = A_1^+ + A_2^+$$

where A_i^+ denotes the pseudo-inverse of A_i.

After finishing this simple one-dimensional example, we return to the general ASM theory. The following theorem covers the case where local preconditioners are used.

Theorem 6.5 *Let $H = H_1 + H_2 + \cdots + H_m$ and $a(u,v) = (Au,v)$. Let $P_i : H \to H_i$ be the orthogonal projection with respect to $a(\cdot,\cdot)$ and let A be symmetric and positive definite. Furthermore, let us assume that the following three conditions are satisfied:*

(1) $\alpha(a(u_1,u_1) + \cdots + a(u_m,u_m)) \le a(u,u)$ *for* $u_1 + \cdots + u_m = u$.

(2) $a(u,u) \le \beta \inf_{u_1+\cdots+u_m=u} \sum_{i=1}^{m} a(u_i,u_i)$.

(3) There are local preconditioners $B_i : H \to H_i$, with $B_i = B_i^*$, such that there exist positive constants c_1 and c_2 such that the spectral equivalence inequalities

$$c_1\,(B_i u,u) \le (Au,u) \le c_2\,(B_i u,u) \quad \forall u \in H_i$$

are fulfilled.

Then, we have

$$\alpha c_1\,(A^{-1}u,u) \le (B^{-1}u,u) \le \beta c_2\,(A^{-1}u,u) \quad \forall u \in H,$$

where $B^{-1} = B_1^+ + \cdots + B_m^+$.

Proof. Note that $P_i = Q_i A_i^{-1} Q_i A$. We have a pseudo-inverse

$$(Q_i A Q_i)^+ = Q_i A_i^{-1} Q_i,$$

since $(Q_i A Q_i) Q_i A_i^{-1} Q_i = Q_i A Q_i A_i^{-1} Q_i = Q_i$. From (1) and (2) we have

$$\alpha(A^{-1}v,v) \le (((Q_1 A Q_1)^+ + \cdots + (Q_m A Q_m)^+)v,v) \le \beta(A^{-1}v,v),$$

and from (3)

$$c_1((Q_i A Q_i)^+u,u) \le (B_i^+ u,u) \le c_2(c(Q_i A Q_i)^+u,u) \quad \forall u \in H_i.$$

Combining the two above inequalities, we get the result of the theorem. □

Remark 6.6 Due to Theorem 6.1, condition (1) is equivalent to $\alpha\, a(u,u) \le a(Pu,u)$, with $P = P_1 + \cdots + P_m$ for all u. Using Theorem 6.2, condition (2) is equivalent to $a(Pu,u) \le \beta\, a(u,u)$ for all u.

In order to prove the stability of the next decompositions the following lemma is required.

Lemma 6.7 *Let $\varphi \in H^{1/2}(-1,0)$ and let us define an extension of φ to $[0,2]$ by the formula*

$$\varphi = \begin{cases} (1-x)\varphi(-x), & x \in [0,1], \\ 0, & x \in [1,2]. \end{cases}$$

Then there exists a constant C such that $\|\varphi\|_{H^{1/2}(-1,2)} \le C\|\varphi\|_{H^{1/2}(-1,0)}$.

Proof. By Lemma 3.9, we have

$$\|\varphi\|^2_{H^{1/2}(-1,2)} \le C_1(\|\varphi\|^2_{H^{1/2}(-1,0)} + \|\varphi\|^2_{H^{1/2}(0,1)} + \|\varphi\|^2_{H^{1/2}(1,2)} + I_1(\varphi) + I_2(\varphi))$$

Note that $\|\varphi\|_{H^{1/2}(1,2)} = 0$. It is trivial to prove

$$\|\varphi\|_{L^2(0,1)} \le \|\varphi\|_{L^2(-1,0)}.$$

Moreover, one obtains

$$\begin{aligned}
|\varphi|^2_{H^{1/2}(0,1)} &= \int_0^1\int_0^1 \frac{|\varphi(-x)(1-x) - \varphi(-y)(1-y)|^2}{|x-y|^2}\,dxdy \\
&\le 2\int_0^1\int_0^1 \frac{|\varphi(-x)(1-x) - \varphi(-x)(1-y)|^2}{|x-y|^2}\,dxdy \\
&\quad +2\int_0^1\int_0^1 \frac{|\varphi(-x) - \varphi(-y)|^2|(1-y)|^2}{|x-y|^2}\,dxdy \\
&\le 2\left(\int_0^1\int_0^1 \frac{|\varphi(-x)(x-y)|^2}{|x-y|^2}\,dxdy + \int_{-1}^0\int_{-1}^0 \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^2}\,dxdy\right) \\
&= 2(|\varphi|^2_{H^{1/2}(-1,0)} + \|\varphi\|^2_{L^2(-1,0)}).
\end{aligned}$$

On the other hand, we have the inequalities

$$\begin{aligned}
I_1(\varphi) = \int_0^1 \frac{(\varphi(-x) - \varphi(-x)(1-x))^2}{x}\,dx &\le \int_0^1 \varphi^2(-x)\frac{x^2}{x}\,dx \le \|\varphi\|^2_{L^2(-1,0)} \\
I_2(\varphi) = \int_0^1 \frac{(\varphi(-x)(1-x))^2}{1-x}\,dx &\le \|\varphi\|^2_{L^2(-1,0)}.
\end{aligned}$$

Gathering all inequalities completes the proof. □

Our goal is to construct the Schur complement on the lines (sub-boundaries) of the interior of the given domain.

We consider a decomposition of Λ into substructures λ_i, i.e. $\Lambda = \bigcup_{i=1}^n \partial\Omega_i = \bigcup_{i=1}^m \lambda_i$. The substructures λ_i, $i = 1, \ldots, m_1$ correspond to cross-points, whereas the substructures λ_i, $i = m_1 + 1, \ldots, m$ correspond to usual lines.

Let us assume that there exists a constant r which is independent of h such that for all $p \in \Lambda$ there exists λ_i:

$$B(p, r) \cap \Lambda \subset \lambda_i,$$

where $B(p, r)$ denotes a ball with a center at p and with a radius r.

Let $H = H_h(\Lambda)$ and $H = H_1 + H_2 + \cdots + H_m$ with

$$H_i = H_h(\lambda_i) = \{\varphi^h \in H_h(\Lambda) | \varphi(x) = 0, \exists x \notin \lambda_i\}.$$

By Lemma 6.7, we have $\forall \varphi^h \in H$, $\exists \varphi_i^h \in H_i$,

$$\|\varphi_1^h\|^2_{\overset{00}{H}{}^{1/2}(\lambda_1)} + \cdots + \|\varphi_m^h\|^2_{\overset{00}{H}{}^{1/2}(\lambda_m)} \le C\|\varphi^h\|^2_{H^{1/2}(\Lambda)}.$$

Let us introduce

$$\tilde{H}_1 = H_1 + \cdots + H_{m_1}, \quad \tilde{H}_2 = H_{m_1+1} + \cdots + H_m.$$

Then we have $H = \tilde{H}_1 + \tilde{H}_2$. So far we have constructed the space satisfying the conditions (1) and (2) in the previous Theorem 6.5. Now, we will construct a preconditioner for the Schur complement by an additive form of pseudo-inverses such as

$$\Sigma^{-1} = \Sigma_1^{+} + \cdots + \Sigma_m^{+}$$

with

$$\Sigma_i = R_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & X^{1/2} & 0 \\ 0 & 0 & 0 \end{bmatrix} R_i^t, \quad i = m_1 + 1, \ldots, m,$$

where R_i is the permutation matrix and X is the matrix corresponding to the one-dimensional Laplacian, i.e.

Note 26: Dots in the wrong place?

$$X = \begin{bmatrix} 2 & -1 & 0 & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & -1 & 2 \end{bmatrix}. \tag{6.2}$$

Hence, we get

$$\Sigma_i^{+} = R_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & X^{-1/2} & 0 \\ 0 & 0 & 0 \end{bmatrix} R_i^t.$$

The preconditioners Σ_i, $i = 1, \ldots, m_1$, are defined later. First we prove the following lemma.

Note 27: Or: $\mathbb{R}^n$ etc. (cf. p. 106)

Lemma 6.8 *Let us assume that the symmetric and positive definite matrices*

$$\Sigma : R^m \to R^m \quad \textit{and} \quad S : R^n \to R^n$$

are given. Let $t : R^m \to R^n$ *such that*

$$\alpha\,(\varphi,\varphi)_\Sigma \leq (t\varphi, t\varphi)_S \leq \beta\,(\varphi,\varphi)_\Sigma \quad \forall \varphi \in R^m,$$

and $(t^\top u, \varphi)_{R^n} = (u, t\varphi)_{R^n}$, *where* $(\cdot,\cdot)_{R^i}$ *denotes the Euclidean inner product. Set* $C = t\Sigma^{-1}t^\top$. *Then we have*

$$\alpha\,(C^{+}u, u) \leq (u,u)_S \leq \beta\,(C^{+}u, u) \quad \forall u \in \mathrm{Im}(t).$$

Proof. By our assumptions, the matrix $(t^\top t)^{-1}$ exists. We note that

$$C^{+} = t(t^\top t)^{-1}\Sigma(t^\top t)^{-1}t^\top,$$

which is easily verified from the following observation

$$C^+C = t(t^\top t)^{-1}\Sigma(t^\top t)^{-1}t^\top(t\Sigma^{-1}t^\top) = t(t^\top t)^{-1}t^\top.$$

Now it is sufficient to check that $t(t^\top t)^{-1}t^\top$ is a projection. If $u \in \text{Im}(t)$, then $u = t\varphi$ for some φ. Thus, we have

$$C^+Cu = t(t^\top t)^{-1}t^\top u = t\varphi = u.$$

Furthermore, for all $v_0 \in (\text{Im}(t))^\perp$, we have

$$0 = (v_0, t\varphi) = (v_0, t(t^\top t)^{-1}t^\top t\varphi) = (t(tTt)^{-1}t^\top v_0, t\varphi) \quad \forall\varphi.$$

Hence, one obtains

$$C^+Cv_0 = 0 \quad \forall v_0 \in (\text{Im}(t))^\perp.$$

Now, for all $u \in \text{Im}(t)$, the relations

$$\begin{aligned}(C^+u, u) &= (C^+t\varphi, t\varphi) = (t(t^\top t)^{-1}\Sigma(t^\top t)^{-1}t^\top t\varphi, t\varphi)\\ &= (t^\top t(t^\top t)^{-1}\Sigma\varphi, t\varphi)\\ &= (\Sigma\varphi, \varphi)\end{aligned}$$

hold. Hence the proof is completed. □

Remark 6.9 In general, $m \leq n$. The operator t can be interpreted as an extension operator.

In the Additive Schwarz Method, we need to define B_i^+. Now we will try to set

$$B_i^+ = (C_i^+)^+ = C_i = t\Sigma^{-1}t^\top,$$

where t is a proper extension operator. The space H_i is defined via $\text{Im}(t) := H_i$. The stability of the decomposition depends on the choice of the extension operators.

7 Additive Schwarz Method on Interfaces

In this section we use the general framework of ASM in order to develop preconditioners on the interfaces.

Let z_0 be a fixed cross point. Let λ be the union of branches emerging from z_0. Let L_i, for $i = 1, \ldots, m$ be each branch and let $L_{m+1} = L_1$ by definition. The trace norm on λ is defined via

$$\|\phi^h\|^2_{H^{1/2}_{00}(\lambda)} = \sum_{i=1}^{m} \|\phi^h\|^2_{H^{1/2}_{00}(L_i \cup L_{i+1})}.$$

Let $x_{i,j}$ be the point on the branch L_i which has the distance jh from z_0. We consider the space decomposition,

$$H_h(\lambda) = H_0 + H_1 + \cdots + H_m,$$

where $H_i = \{\phi^h \in H_h(\lambda) | \phi^h(x) = 0, x \notin L_i\}$ and $H_0 = \{\phi^h \in H^h(\lambda) | \phi^h(x_{1,j}) = \cdots = \phi^h(x_{m,j}), j = 1, 2, \ldots, k\}$. Here we assume that each L_i has the same number k of nodes.

Lemma 7.1 *There exists a constant c independent of h such that, for each $\phi^h \in H^h(\lambda)$, there exist $\phi_i^h \in H_i$ with $\sum_{i=0}^m \phi_i^h = \phi^h$ which satisfies*

$$\|\phi_0^h\|^2_{H_{00}^{1/2}(\lambda)} + \|\phi_1^h\|^2_{H_{00}^{1/2}(\lambda)} + \cdots + \|\phi_m^h\|^2_{H_{00}^{1/2}(\lambda)} \leq c\|\phi^h\|^2_{H_{00}^{1/2}(\lambda)}.$$

Proof. Let $\phi \in H_h(\lambda)$. First, we set $\phi_0^h(x_{i,j}) = \phi^h(x_{1,j})$, for $j = 1, \ldots, k$ and $i = 1, \ldots, m$, i.e. we take the values of the first branch on the another branches. Due to our definition, we have $\phi_0^h \in H_0$. Let $\psi^h = \phi^h|_{L_1}$. Since

$$\|\phi_0^h\|^2_{H_{00}^{1/2}(\lambda)} = m\|\phi^h\|^2_{H_{00}^{1/2}(L_1 \cup L_2)} \simeq \|\psi^h\|^2_{\check{H}^{1/2}(L_1)} \simeq (\Sigma_{ND}\psi, \psi),$$

there exists a constant c_1 which is independent of h such that

$$\|\phi_0^h\|^2_{H_{00}^{1/2}(\lambda)} \leq c_1\|\phi^h\|^2_{H_{00}^{1/2}(\lambda)}.$$

Let $\xi^h = \phi^h - \phi_0^h$. Next, we define the hat functions ϕ_i^h, $i \geq 1$ via $\phi_i^h(x_{i,j}) = \xi^h(x_{i,j})$. Then, we have

$$\|\xi^h\|^2_{H_{00}^{1/2}(\lambda)} \leq c_2\|\phi^h\|^2_{H_{00}^{1/2}(\lambda)}.$$

This implies the estimates

$$\|\phi_i^h\|^2_{H_{00}^{1/2}(\lambda)} \simeq \|\phi_i^h\|^2_{H_{00}^{1/2}(L_i)} \simeq (\Sigma_{DD}\phi_i, \phi_i).$$

Hence we obtain

$$\|\phi_i^h\|^2_{H_{00}^{1/2}(\lambda)} \leq \|\xi^h\|^2_{H_{00}^{1/2}(\lambda)}.$$

Continuing the above processes, we can prove the lemma. □

The lemma shows that

$$\frac{1}{c}a(\phi^h, \phi^h) \leq a((P_0 + \cdots + P_m)\phi^h, \phi^h) \leq (m+1)a(\phi^h, \phi^h).$$

Let t be the extension operator such that, for each $\psi^h = \begin{bmatrix}\psi_0 & \psi_1 & \cdots & \psi_m\end{bmatrix}^\top$, the relation $t\phi^h = \begin{bmatrix}\psi_0 & \eta & \cdots & \eta\end{bmatrix}^\top$ holds, where $\eta = \begin{bmatrix}\psi_1 & \cdots & \psi_m\end{bmatrix}^\top$. Then, we have the relations $H_0 = t \cdot F$, $F = H_h(L_1)$ and

$$\|\psi^h\|_{\check{H}^{1/2}(L_1)} \leq \|t\psi^h\|_{H_{00}^{1/2}(\lambda)} \leq C\|\psi^h\|_{\check{H}^{1/2}(L_1)}.$$

This gives

$$B_i^+ = t\Sigma_{ND}^{-1}t^\top.$$

Now, the whole interface space $H_h(\Lambda)$ is decomposed into subspaces. Let

$$H_h(\Lambda) = H_1^{(N)} + \cdots + H_{m_1}^{(N)} + H_{m_1+1}^{(0)} + \cdots + H_m^{(0)}.$$

The spaces $H_i^{(N)}$, $i = 1, \ldots, m_1$, are the subspaces corresponding to cross points, whereas the spaces

$$H_i^N = \{\varphi^h \in H_h(\Lambda) \mid \varphi^h(x) = t_i\psi^h(x), x \in \lambda_i, \varphi^h(x) = 0, x \notin \lambda_i\}$$

and $H_i^{(0)}$, $i = m_1 + 1, \ldots, m$, are the subspaces corresponding to intervals between cross points, i.e.

$$H_i^0 = \{\varphi^h \in H_h(\Lambda) \mid \varphi^h(x) = 0, x \notin \lambda_i\}.$$

Let

$$B^{-1} = B_{N,1}^+ + \cdots + B_{N,m_1}^+ + B_{0,m_1+1}^+ + \cdots + B_{0,m}^+, \tag{7.1}$$

where

$$B_{0,i}^+ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & X^{-1/2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with the one-dimensional Laplacian operator X, cf. (6.2) and

$$B_{N,i}^+ = t_i \Sigma_{ND}^{-1} t_i^\top.$$

Then, the matrix B (7.1) is a preconditioner for the Schur complement $S = A_0 - \sum_{i=1}^n A_{0i}A_i^{-1}A_{i0}$. However, the direct computation of A_i^{-1} is too expensive. In order to replace A_i^{-1} by a local preconditioner on Ω_i, we have to use ASM. This is presented in the following section.

7.1 Inexact Solvers

In this section, we consider a nonoverlapping partition of Ω into subdomains Ω_i, i.e. $\overline{\Omega} = \bigcup_{i=1}^n \overline{\Omega_i}$, $\Omega_i \cap \Omega_j = \emptyset$, if $i \neq j$. The space decomposition $H_h(\Omega) = H_0 + H_1$ is considered, where

$$\begin{aligned} H_0 &= H_{h,0}(\Omega_1) \oplus \cdots \oplus H_{h,0}(\Omega_n) \quad \text{and} \\ H_{h,0}(\Omega_i) &= \{\phi^h \in H_h(\Omega) | \phi^h(x) = 0, x \notin \Omega_i\} \end{aligned}$$

correspond to the Dirichlet problems on the subdomains Ω_i. The space H_1 is defined below.

Theorem 7.2 *In addition to the above assumptions, let us assume the following:*

(A) There exist operators B_i which satisfy

$$c_1\|u^h\|_{H^1(\Omega)}^2 \leq (B_i u, u) \leq c_2\|u^h\|_{H^1(\Omega)}^2 \, \forall u^h \in H_{h,0}(\Omega_i), \quad i = 1, \ldots, n.$$

(B) There exist extension operators $t_i : H_h(\Gamma_i) \to H_h(\Omega_i)$ *such that the inequalities*

$$\|t_i\phi^h\|_{H_1(\Omega_i)} \le c_3\|\phi^h\|_{H^{1/2}(\Gamma_i)}$$

hold for $i = 1, \ldots, n$. *Let* $H_1 = tH_h(\lambda)$*, where t now denotes the global extension operator composed of the local ones.*

(C) There exists an operator Σ *with*

$$c_4\|\phi^h\|^2_{H^{1/2}(\lambda)} \le (\Sigma\phi, \phi) \le c_5\|\phi^h\|^2_{H^{1/2}(\Lambda)}\, \forall\phi^h \in H_h(\Lambda).$$

Set

$$B^{-1} = \begin{bmatrix} 0 & & & \\ & B_1^{-1} & & \\ & & \ddots & \\ & & & B_n^{-1} \end{bmatrix} + t\Sigma^{-1}t^\top. \tag{7.2}$$

Then there exist constants α *and* β *which only depend on* $c_1, c_2, \ldots, c_5$ *such that*

$$\alpha\,(Bu, u) \le (Au, u) \le \beta\,(Bu, u) \qquad \forall u \in H_h(\Omega).$$

Proof. Let $u^h \in H_h(\Omega)$ denote the trace function of u^h by $\phi^h \in H^h(\Lambda)$, i.e. $\phi^h(x) = u^h(x), x \in \Lambda$. By the trace theorem, there exists a constant c_6 independent of h such that

$$\|\phi^h\|_{H^{1/2}(\Lambda)} \le c_6\|u^h\|^2_{H^1(\Omega)}. \tag{7.3}$$

Let $u_1^h = t\phi^h$. Using (B) and (7.3), we have

$$\|u_1^h\|^2_{H^1(\Omega)} \le c_3\|\phi^h\|_{H^{1/2}(\Lambda)} \le c_3c_6\|u^h\|^2_{H^1(\Omega)}.$$

Let $u_0^h = u^h - u_1^h$. Using the triangle inequality and ASM proves the assertion. □

7.2 Explicit Extension Operators

The definition of the preconditioner (7.2) requires an extension operator t which satisfies (B). This section is devoted to the construction of the extension operator t. The simplest choice of an extension operator which satisfies (B) is the harmonic extension. However, in every preconditioning step a problem for the Laplacian has to be solved. This is too expensive. Therefore, we have to find another one.

Let (s, n) be a near boundary coordinate system, where s denotes the tangential and n denotes the normal coordinate. Let ϕ be a given function defined on the boundary Γ of a domain Ω. For the continuous case, we can define the extended function $u = t\phi$ by

$$u(s, n) = \xi(n)\frac{1}{n}\int_s^{s+n} \phi(t)dt,$$

where $\xi(n) = 1 - \frac{n}{D}$, and D is the thickness of the near boundary strip. Let ϕ be some finite element function which is given by its nodal values $\phi(l)$, $l = 0, \ldots, i + j$.

The function u is defined by its values at the nodal points z_{ij}, where the first index corresponds to the tangential component and the second to the normal component, i.e. z_{i0} are the nodes on the boundary. Moreover, we introduce D_{ij} as the cells of an auxiliary grid. Then, the extension u is defined by the following three steps.

1. $V(z_{ij}) = \sum_{l=0}^{j} \phi(i+l)$,
2. $U(z_{ij}) = \frac{1-\frac{j}{n}}{j+1} V(z_{ij})$,
3. $u^h(z_l) = \begin{cases} U(z_{ij}) & \text{if } z_l \in D_{ij} \\ 0 & \text{if } z_l \notin \bigcup_{ij} D_{ij} \end{cases}$

In matrix vector-notation, we have $u = t\phi = P_3 P_2 P_1 \phi$, where the matrix P_3 is given by

$$P_3 = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ & I & \end{bmatrix}.$$

Note that the lower identity matrix corresponds to D. Then we have

$$\|u^h\|_{H_h^1(\Omega^h)} \leq c\|V\|_{H_h^1(D^h)}.$$

Due to the second step of the extension, the matrix P_2 is a diagonal matrix, i.e. $P_2 = \text{diag}\{\ldots, \frac{1-j/n}{j+1}, \ldots\}$. The matrix P_1 corresponds to the first step and has the form

$$P_1 = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ 1 & . & . & 1 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

The identity block corresponds to $V(z_{i,0}) = \phi(i)$, $i = 0, \ldots, N-1$, whereas the values of the j-th layer in normal direction are given by $P_1\phi = V(z_{i,j+1}) = V(z_{i,j}) + \phi(i+j+1)$, $0 \leq i \leq N-1$, $0 \leq j \leq M$. The total computational cost for the multiplication with P_2 is $O(h^{-2})$.

Let $t^\top = P_1^\top P_2^\top P_3^\top$ be the adjoint operator. For any given function V, we define the function W by

$$W(z_{i,M}) = V(z_{i,M}), \quad i = 0, \ldots, M$$

and for $i = 0, \ldots, N$ and $j = M, M-1, \ldots, 1$, let

$$W(z_{i,j-1}) = W(z_{i,j}) + V(z_{i,j-1}).$$

If we set $\phi_i = \sum_{j=0}^{N} \sum_{l=i-j}^{i} V(z_{l,j})$, and $W(z_{i,M}) = V(z_{i,M})$

Note 28: "and" before displayed formula?

$$W_1(t_{i,j-1}) = W_1(z_{i,j}) + V(z_{i,j-1})$$

we arrive at $\phi_i = \sum_{j=0}^{M} W(z_{i-j})$. Therefore the total cost for a multiplication with $t^\top$ is again $O(h^{-2})$.

Remark 7.3 We note that multilevel (hierarchical) explicit extension operators were suggested in [10]. This multilevel decomposition on a boundary of some subdomain is based on the results of [45]. But this method is not asymptotically optimal. An optimal method (with respect to the arithmetical coast and the norms of the extension operators) of the multilevel explicit extension of functions was suggested in [31] and [11]. In this case the multilevel decomposition on the boundary of the subdomain is based on BPX-like decompositions.

8 Domain Decomposition with Many Subdomains

In order to use parallel computers with many processors a decomposition of an original domain into many subdomains ($n \gg 1$) of small measure is required.

Let Ω be a domain of diameter $O(1)$ with boundary Γ, and set

$$\Omega_\varepsilon = \{(x, y) : x = \varepsilon s,\ y = \varepsilon t,\ (x, y) \in \Omega\}$$

with boundary Γ_ε. Here we present some results on the trace theory in general Sobolev spaces with small diameters which are characterized by a small parameter ε. For the FEM solution of elliptic and parabolic problems this parameter ε usually is equivalent to a mesh size of the coarse grid H or the average diameter H of the subdomains.

Lemma 8.1 *There exists $c_1 \neq c_1(\varepsilon)$ such that, for all $u \in H^1(\Omega_\varepsilon)$, we have*

$$\varphi(x) = u(x),\ x \in \Gamma_\varepsilon, \qquad |\varphi|_{H^{1/2}(\Gamma_\varepsilon)} \leq c_1 |u|_{H^1(\Omega_\varepsilon)},$$

and vice versa there exists $c_2 \neq c_2(\varepsilon)$ such that for every $\varphi \in H^{1/2}(\Gamma_\varepsilon)$, there exists $u \in H^1(\Omega_\varepsilon)$ satisfying

$$\varphi(x) = u(x),\ x \in \Gamma_\varepsilon, \qquad |u|_{H^1(\Omega_\varepsilon)} \leq c_2 |\varphi|_{H^{1/2}(\Gamma_\varepsilon)}.$$

Proof. Using the change of variables, i.e. $(x \to s), (y \to t)$, we have

$$|u|^2_{H^1(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \int_{\Omega} \left(\frac{\partial \tilde{u}}{\partial s}\right)^2 + \left(\frac{\partial \tilde{u}}{\partial t}\right)^2 = |\tilde{u}|^2_{H^1(\Omega)}$$

and

$$\begin{aligned} |\varphi|^2_{H^{1/2}(\Gamma_\varepsilon)} &= \int_{\Gamma_\varepsilon}\int_{\Gamma_\varepsilon} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^2}\,dxdy \\ &= \int_{\Gamma}\int_{\Gamma} \frac{(\tilde{\varphi}(s)-\tilde{\varphi}(t))^2}{|s-t|^2}\,dsdt \;=\; |\tilde{\varphi}|^2_{H^{1/2}(\Gamma)} \end{aligned}$$

which prove the lemma. □

Now we define

$$\|\varphi\|^2_{H^{1/2}_\varepsilon(\Gamma_\varepsilon)} = \varepsilon\|\varphi\|^2_{L_2(\Gamma_\varepsilon)} + |\varphi|^2_{H^{1/2}(\Gamma_\varepsilon)}.$$

Lemma 8.2 *There exists $c_1 \neq c_1(\varepsilon)$ such that for all $u \in H^1(\Omega_\varepsilon)$,*

$$\varphi(x) = u(x),\; x \in \Gamma_\varepsilon, \qquad \|\varphi\|_{H^{1/2}_\varepsilon(\Gamma_\varepsilon)} \le c_1\|u\|_{H^1(\Omega_\varepsilon)}.$$

There exists $c_2 \neq c_2(\varepsilon)$ such that for every $\varphi \in H^{1/2}(\Gamma_\varepsilon)$, there exists $u \in H^1(\Omega_\varepsilon)$ satisfying

$$\varphi(x) = u(x),\; x \in \Gamma_\varepsilon, \qquad \|u\|_{H^1(\Omega_\varepsilon)} \le c_2\|\varphi\|_{H^{1/2}_\varepsilon(\Gamma_\varepsilon)}.$$

Proof. Using evident transformations we have

$$\|u\|^2_{H^1(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} u^2 + \int_{\Omega_\varepsilon} |\nabla u|^2 = \varepsilon^2\|\tilde{u}\|^2_{L_2(\Omega)} + |\tilde{u}|^2_{H^1(\Omega)} \approx \varepsilon^2\|\tilde{\varphi}\|^2_{L_2(\Gamma)} + |\tilde{u}|^2_{H^1(\Omega)},$$

$$\varepsilon\|\varphi\|^2_{L_2(\Gamma_\varepsilon)} + |\varphi|^2_{H^{1/2}(\Gamma_\varepsilon)} = \varepsilon^2\|\tilde{\varphi}\|^2_{L_2(\Gamma)} + |\varphi|^2_{H^{1/2}(\Gamma)}.$$

□

Lemma 8.3 *There exists $c_1 \neq c_1(\varepsilon)$ such that if $\int_{\Gamma'_\varepsilon} \varphi(x)\,dx = 0$, $\mathrm{meas}(\Gamma'_\varepsilon) \approx \varepsilon$, then*

$$\frac{1}{\varepsilon}\|\varphi\|^2_{L_2(\Gamma_\varepsilon)} + |\varphi|^2_{H^{1/2}(\Gamma_\varepsilon)} \le c_1|\varphi|^2_{H^{1/2}(\Gamma_\varepsilon)}.$$

Proof. In order to estimate $\frac{1}{\varepsilon}\|\varphi\|^2_{L_2(\Gamma_\varepsilon)}$, we use the following simple manipulations

$$\begin{aligned} \frac{1}{\varepsilon}\|\varphi\|^2_{L_2(\Gamma_\varepsilon)} + |\varphi|^2_{H^{1/2}(\Gamma_\varepsilon)} &= \|\tilde{\varphi}\|^2_{L_2(\Gamma)} + |\tilde{\varphi}|^2_{H^{1/2}(\Gamma)} \\ &\le c_2\|\tilde{u}\|^2_{H^1(\Omega)} \quad \text{(Theorem 3.5)} \\ &\le c_3|\tilde{u}|^2_{H^1(\Omega)} \\ &\le c_1|\varphi|^2_{H^{1/2}(\Gamma_\varepsilon)}. \end{aligned}$$

This completes the proof of the lemma. □

Let $t : H^{1/2}(\Gamma_\varepsilon) \to H^1(\Omega_\varepsilon)$ be given by

$$u = t\varphi = \xi v, \qquad \xi(n) = 1 - \frac{n}{D},$$

where $\xi(n)$ is defined above, $v \in H^1(\Omega_\varepsilon)$ is a preserving of norm, but does not satisfy to a homogeneous Dirichlet boundary condition. The cut-off function ξ gives to u the homogeneous Dirichlet boundary condition. Then

Note 29: Correct: is a preserving of norm … satisfy to a

$$|u|^2_{H^1(\Omega_\varepsilon)} \cong |\xi'|^2 \|v\|^2_{L_2(\Omega_\varepsilon)} + |\xi|\, |v|^2_{H^1(\Omega_\varepsilon)}, \qquad |\xi'| = \frac{1}{\varepsilon}.$$

Since the $|\xi'|$ can be big, we suggest the following construction of the extension operator. For $\varphi \in H^{1/2}(\Gamma_\varepsilon)$, let $\varphi = \varphi_0 + \varphi_1$, where $\varphi_0 \equiv$ constant and $\int_{\Gamma_\varepsilon} \varphi_1(x)\, dx = 0$. Now we define $u_0 \equiv$ constant $= \varphi_0$ and $u_1 = t\varphi_1 = \xi v$. Then we have the inequalities

$$\begin{aligned} \|u_0\|_{L_2(\Omega_\varepsilon)} &\le c_2 \varepsilon \|\varphi_0\|_{L_2(\Gamma_\varepsilon)}, \\ \left(\frac{1}{\varepsilon}\right)^2 \|v\|^2_{L_2(\Omega_\varepsilon)} + |v|^2_{H^1(\Omega_\varepsilon)} &\le c_3 |\varphi_1|^2_{H^{1/2}(\Gamma_\varepsilon)} = c_3 |\varphi|^2_{H^{1/2}(\Gamma_\varepsilon)}. \end{aligned}$$

Using Lemma 6.7 we can easily prove the following lemma.

Lemma 8.4 *For a given function $\phi : [-1, 0] \mapsto R$, let*

$$\varphi(x) = \begin{cases} (1 - x)\varphi(-x), & x \in [0, 1], \\ 0, & x \in [1, 2]. \end{cases}$$

Then there exists a constant c such that $\|\varphi\|_{H^{-1/2}(-1,2)} \le c \|\varphi\|_{H^{1/2}(-1,0)}$.

Note 30: Minus in $H^{-1/2}$ correct?

Lemma 8.5 *For $\varphi \in H^{1/2}(0, 3\varepsilon)$, we have ($c \ne c(\varepsilon)$)*

$$\frac{1}{\varepsilon} \|\varphi\|^2_{L_2(0,3\varepsilon)} + |\varphi|^2_{H^{1/2}(0,3\varepsilon)} \le c \|\varphi\|^2_{H^{1/2}(0,3\varepsilon)}.$$

Let $\varphi = \varphi_1 + \varphi_2$ with

$$\varphi(x) = \begin{cases} \varphi_1(x), & x \in (0, \varepsilon), \\ \varphi_2(x), & x \in (2\varepsilon, 3\varepsilon), \end{cases}$$

where φ_1 and φ_2 are defined on $[0, 3\varepsilon]$ according to Lemma 8.4. Then

$$\frac{1}{\varepsilon} \|\varphi_1\|^2_{L_2(0,3\varepsilon)} + |\varphi_1|^2_{H^{1/2}(0,3\varepsilon)} + \frac{1}{\varepsilon} \|\varphi_2\|^2_{L_2(0,3\varepsilon)} + |\varphi_2|^2_{H^{1/2}(0,3\varepsilon)} \le c_1 \|\varphi\|^2_{H^{1/2}(0,3\varepsilon)}.$$

Proof. It is easy to see that

$$\int_0^{3\varepsilon} \int_0^{3\varepsilon} \frac{(\varphi(x) - \varphi(y))^2}{|x - y|^2}\, dx dy = \int_0^3 \int_0^3 \frac{(\tilde\varphi(s) - \tilde\varphi(t))^2}{|s - t|^2}\, ds dt$$

$$\frac{1}{\varepsilon} \int_0^{3\varepsilon} \varphi^2(x)\, dx = \int_0^3 \tilde\varphi(s)^2\, ds.$$

□

Lemma 8.6 *Let $\varphi^\varepsilon \in H^{1/2}(0,3\epsilon)$ be a continuous and piecewise linear function with $\varphi^\varepsilon(i\varepsilon) = \varphi_i$, and linear on the intervals $[i\varepsilon,(i+1)\varepsilon]$, $i = 0,\ldots,3$. Then*

Note 31: twice linear correct?

$$\|\varphi^\varepsilon\|^2_{H^{1/2}_\varepsilon(0,3\varepsilon)} \approx \sum_{i=0}^{3} \varepsilon^2\varphi_i^2 + \sum_{i=0}^{3}\sum_{j=0}^{3}(\varphi_i - \varphi_j)^2.$$

Proof. The assertion immediately follows from the relations

$$\varepsilon\|\varphi^\varepsilon\|^2_{L_2(0,3\varepsilon)} \approx \sum_{i=0}^{3}\varepsilon^2\varphi_i^2, \quad \text{and} \quad |\varphi^\varepsilon|^2_{H^{1/2}(0,3\varepsilon)} \approx \sum_{i=0}^{3}\sum_{j=0}^{3}(\varphi_i-\varphi_j)^2.$$

□

In order to construct asymptotically optimal decompositions in the case of many subdomains, we have to consider the so-called coarse subspace beside the local subspaces. To do it, we use the following lemma.

Lemma 8.7 *There exists $c \neq c(h,\varepsilon)$ such that, for every $\varphi^h \in H_h(0,3\varepsilon)$, there are $\varphi^\varepsilon, \varphi_1^h, \varphi_2^h$ satisfying*

$$\begin{aligned}
&\varphi^h = \varphi^\varepsilon + \varphi_1^h + \varphi_2^h, && \varphi^\varepsilon \textit{ piecewise linear},\\
&\varphi_1^h(x) = 0, && x \in (2\varepsilon, 3\varepsilon),\\
&\varphi_2^h(x) = 0, && x \in (0,\varepsilon),
\end{aligned}$$

and

$$\|\varphi^\varepsilon\|^2_{H^{1/2}(0,3\varepsilon)} + \|\varphi_1^h\|^2_{H^{1/2}(0,3\varepsilon)} + \|\varphi_2^h\|^2_{H^{1/2}(0,3\varepsilon)} \le c\|\varphi^h\|^2_{H^{1/2}(0,3\varepsilon)}.$$

Proof. We define the piecewise linear function φ^ε by the values

$$\begin{aligned}
\varphi_0 = \varphi_1 &= \frac{1}{\varepsilon}\int_0^\varepsilon \varphi^h(x)\,dx,\\
\varphi_2 = \varphi_3 &= \frac{1}{\varepsilon}\int_{2\varepsilon}^{3\varepsilon} \varphi^h(x)\,dx.
\end{aligned}$$

Then we arrive at the following estimates:

$$\begin{aligned}
(\varphi_i)^2 &= \left(\frac{1}{\varepsilon}\int_{x_i}^{x_{i+1}} \varphi(x)\,dx\right)^2 \le \frac{1}{\varepsilon^2}\varepsilon\int_{x_i}^{x_{i+1}} \varphi^2(x)\,dx,\\
\sum_{i=0}^{3}\varepsilon^2(\varphi_i)^2 &\le \varepsilon\|\varphi\|^2_{L_2(0,3\varepsilon)},\\
(\varphi_i-\varphi_j)^2 &= \left(\frac{1}{\varepsilon}\int_{x_i}^{x_{i+1}} \varphi^h(x)\,dx - \frac{1}{\varepsilon}\int_{x_j}^{x_{j+1}} \varphi^h(x)\,dx\right)^2\\
&= \frac{1}{\varepsilon^2}\left(\frac{1}{\varepsilon}\int_{x_i}^{x_{i+1}}\int_{x_j}^{x_{j+1}} \varphi^h(x)\,dydx - \frac{1}{\varepsilon}\int_{x_i}^{x_{i+1}}\int_{x_j}^{x_{j+1}} \varphi^h(x)\,dxdy\right)^2\\
&\le \frac{4}{\varepsilon^2}\left(\int_{x_i}^{x_{i+1}}\int_{x_j}^{x_{j+1}} \frac{\varphi^h(x)-\varphi^h(y)}{|x-y|}\,dxdy\right)^2\\
&\le 4\int_{x_i}^{x_{i+1}}\int_{x_j}^{x_{j+1}} \frac{(\varphi^h(x)-\varphi^h(y))^2}{|x-y|^2}\,dxdy,\\
\psi^h &= \varphi^h-\varphi^\varepsilon,\\
\int_0^\varepsilon \psi^h(x)\,dx &= \int_{2\varepsilon}^{3\varepsilon}\psi^h(x)\,dx = 0.
\end{aligned}$$

This completes the proof of the lemma. □

Let us define the substructure as above. Then the following lemma holds.

Lemma 8.8 *Let* $\overline{\Omega} = \bigcup_{i=1}^n \overline{\Omega_i}$*, where* Ω_i *is polygonal and* $\operatorname{diam}\Omega_i = O(H)$*, and let* $\Lambda = \bigcup_{i=1}^m \lambda_i$*. Then there exists* $c \ne c(h,H)$ *such that, for every* $\varphi^h \in H_h(\Lambda)$*, there are* $\varphi^H, \varphi_1^h, \ldots, \varphi_m^h$ *satisfying*

(i) φ^H *piecewise linear on the coarse grid* $\bigcup_{i=1}^n \partial\Omega_i$*, and*

(ii) $\varphi_i^h(x) = 0$*, if* x *is a cross point of* $\lambda_i,\ i = 1,\ldots,m$*.*

Then we have

$$\|\varphi^H\|_{H^{1/2}(\Lambda)} \le c_1\|\varphi^h\|_{H^{1/2}(\Lambda)},$$

$$\sum_{i=1}^m \|\varphi_i^h\|^2_{H^{1/2}(\Lambda)} \le C_1\|\varphi^h\|^2_{H^{1/2}(\Lambda)},$$

$$\Sigma^{-1} = \Sigma_H^+ + \Sigma_1^+ + \cdots + \Sigma_m^+, \qquad (\Sigma_i\varphi,\varphi) \simeq \|\varphi^h\|_{\underset{H}{00}{}^{1/2}(\lambda_i)},$$

$$(\Sigma_H\varphi,\varphi) = H^2\sum\varphi_i^2 + \sum_i\sum_j(\varphi_i-\varphi_j)^2,$$

$$(\Sigma\varphi,\varphi) \simeq \|\varphi^h\|_{H^{1/2}(\Lambda)}.$$

Proof. The proof of this lemma follows from the general theory of ASM and the previous results. □

9 Additive Schwarz Method (ASM) and Multilevel Decomposition

The ASM preconditioner also requires preconditioners for the subdomains. This section presents a possible preconditioner. Let Ω be a domain in R^2, $\Omega_i, i = 1, \ldots, n$ be a disjoint subdomain of Ω and $\Lambda = \bigcup_{i=1}^{n} \partial\Omega_i$. In addition, we introduce matrices B_i which define equivalent norms for $-\Delta_{\Omega_i}$ and a matrix Σ which generates an equivalent norm on the space $H^{1/2}(\Lambda)$, i.e. $B_i \longleftrightarrow -\Delta_{\Omega_i}$ and $\Sigma \longleftrightarrow H^{1/2}(\Lambda)$, respectively.

Let $t\colon H^{1/2}(\Lambda) \to H(\Omega)$ be an extension operator. Then the inexact ASM preconditioner (7.2) has the form

$$B^{-1} = \begin{bmatrix} 0 & & & \\ & B_1^{-1} & & \\ & & \ddots & \\ & & & B_n^{-1} \end{bmatrix} + t\Sigma^{-1}t^{\top},$$

where the first block corresponds to Λ and the block $i+1$ to Ω_i. Let us fix the subdomain Ω_i and omit the index i. First we consider the case when Ω is polygonal. Let $\Omega_0^h, \Omega_1^h, \ldots, \Omega_J^h$ be a sequence of grids on Ω and $W_0 \subset W_1 \subset \cdots \subset W_J = W$ be a sequence of nested spaces, respectively. We denote the nodal basis in W_k by $\{\phi_i^{(k)}\}_{i=1,2,\ldots,n_k}$, and $\Phi_i^{(k)} = \{\alpha \cdot \phi_i^{(k)} | \alpha \in R\}$ the vector space spanned by the basis function $\phi_i^{(k)}$. Then we obviously have the representations

$$W_k = \Phi_1^{(k)} + \cdots + \Phi_{n_k}^{(k)} \quad \text{and} \quad W = \sum_{k=0}^{J} \sum_{i=1}^{n_k} \Phi_i^{(k)}. \tag{9.1}$$

Let $P_i^{(k)} : W \to \Phi_i^{(k)}$ be an orthogonal projection with respect to $a(\cdot, \cdot)$.

The so-called BPX preconditioner (multilevel preconditioner), which corresponds to the space decomposition (9.1), was proposed by J. H. Bramble, J. E. Pasciak and J. Xu in [5]. The investigation of the optimality of the BPX preconditioner is due to P. Oswald [33], see also [3, 7, 34, 46]. The BPX preconditioner can also be considered as ASM for a special decomposition of the original finite element space into subspaces.

Due to the space decomposition (9.1), the following result can be shown.

Theorem 9.1 *There exist two constants α and β, which are independent of h, such that*

(1) For every $u^h \in W$, there exist $u_i^{(k)} \in \Phi_i^{(k)}$ such that

$$\sum_{k=0}^{J}\sum_{i=1}^{n_k} u_i^{(k)} = u^h \tag{9.2}$$

with

$$\sum_{k=0}^{J}\sum_{i=1}^{n_k} \|u_i^{(k)}\|^2_{H^1(\Omega)} \le \frac{1}{\alpha}\|u^h\|_{H^1(\Omega)}. \tag{9.3}$$

(2) Moreover, the inequality

$$\|u^h\|^2_{H^1(\Omega)} \le \beta \inf_{\sum_{k=0}^{J}\sum_{i=1}^{n_k} v_i^{(k)}=u^h} \|v_i^{(k)}\|^2_{H^1(\Omega)}, \quad v_i^{(k)} \in \Phi_i^{(k)} \tag{9.4}$$

holds.

Remark 9.2 Note that relation (9.3) implies the estimate

$$\alpha\,\|u^h\|^2_{H^1(\Omega} \le a\left(\sum_{k=0}^{J}\sum_{i=1}^{n_k} P_i^{(k)} u^h, u^h\right) \le \beta\,(u^h, u^h).$$

The proof uses the following fundamental result that was already suggested by P. L. Butzer and K. Scherer in [6].

Lemma 9.3 *Let $Q_k : W \to W_k$ be the orthogonal projection in $L_2(\Omega)$. Then there exist two constants C_1 and C_2, which are independent of h and J, such that*

$$\begin{aligned} C_1\|u^h\|^2_{H^1(\Omega)} \;\le\; |||h^h||| \;&:=\; \|Q_0 u^h\|^2_{L_2(\Omega)} + \sum_{k=1}^{J} h_k^{-2}\|(Q_k - Q_{k-1})u^h\|^2_{L_2(\Omega)} \\ &\le\; C_2\|u^h\|^2_{H^1(\Omega)} \end{aligned}$$

and

$$C_1|||u^h||| \;\le\; \inf_{\substack{u^h=u_0^h+\cdots+u_J^h \\ u_k^h\in W_k}} \sum_{k=0}^{J} h_k^{-2}\|u_k^h\|^2_{L_2(\Omega)} \;\le\; C_2|||u^h|||.$$

For the finite element case, the rigorous proof of the lemma can be found in [34].

Proof (Theorem 9.1). The proof of (9.3) considers the decomposition

$$u^h = Q_0 u^h + \sum_{k=1}^{J}(Q_k - Q_{k-1})u^h = v_0^h + v_1^h + \cdots + v_J^h,$$

where $v_k^h \in W_k$. Note that $Q_J u^h = u^h$. On the other hand, the fact that $v_k^h \in W_k$ implies the relation

$$v_k^h = \sum_{i=1}^{n_k} \alpha_i^{(k)}\phi_i^{(k)} = \sum_{i=1}^{n_k} v_i^{(k)},$$

where $v_i^{(k)} \in \Phi_i^{(k)}$. Thus, we have

$$u^h = \sum_{k=0}^{J} \sum_{i=1}^{n_k} v_i^{(k)}.$$

Using the inverse inequality, Lemma 9.3 and the fact $v_k^h = (Q_k - Q_{k-1})u^h$, we obtain

$$\begin{aligned} \sum_{k=0}^{J} \sum_{i=0}^{n_k} \|v_i^{(k)}\|^2_{H^1(\Omega)} &\approx \sum_{k=0}^{J} \sum_{i=0}^{n_k} h_k^{-2} \|v_i^{(k)}\|^2_{L_2(\Omega)} \\ &\approx \sum_{k=0}^{J} h_k^{-2} \|v_k^h\|^2_{L_2(\Omega)} \\ &\leq \|u^h\|^2_{H^1(\Omega)}. \end{aligned}$$

This proves (9.3). In order to prove (9.4), we start with

$$\|u^h\|^2_{H^1(\Omega)} \leq \beta \inf \sum_{k=0}^{J} \sum_{i=1}^{n_k} h_k^{-2} \|v_i^{(k)}\|^2_{L_2(\Omega)}.$$

Moreover,

$$\begin{aligned} \inf_{\substack{v_i^{(k)} \in \Phi_i^{(k)} \\ \sum_{k=0}^{J} \sum_{i=1}^{n_k} v_i^{(k)} = u^h}} \sum_{k=0}^{J} \sum_{i=1}^{n_k} h_k^{-2} \|v_i^{(k)}\|^2_{L_2(\Omega)} &= \inf_{\alpha_i^{(k)}} \sum_{k=0}^{J} \sum_{i=1}^{n_k} h_k^{-2} \|\alpha_i^{(k)} \phi_i^{(k)}\|^2_{L_2(\Omega)} \\ &\geq C \inf_{\alpha_i^{(k)}} \sum_{k=0}^{J} h_k^{-2} \|v_k^h\|^2_{L_2(\Omega)} \\ &= C \inf_{v_k^h \in W_k} \sum_{k=0}^{J} \|v_k^h\|^2_{L_2(\Omega)} \\ &\geq C \cdot C_1 \|u^h\|^2_{H^1(\Omega)}. \end{aligned}$$

This completes the proof of Theorem 9.1. □

Let us give an example of the above theorem. Let $A_i^{(k)} : \Phi_i^{(k)} \to \Phi_i^{(k)}$. Let us define the L_2 orthogonal projection $Q_i^{(k)} : W \to \Phi_i^{(k)}$ as follows:

$$Q_i^{(k)} u^h = \frac{(u^h, \phi_i^{(k)})_{L_2(\Omega)}}{(\phi_i^{(k)}, \phi_i^{(k)})_{L_2(\Omega)}} \phi_i^{(k)}.$$

We define $P_i^{(k)} : W \to \Phi_i^{(k)}$ by setting

$$P_i^{(k)} = (A_i^{(k)})^{-1} Q_i^{(k)}$$

and $a(\cdot,\cdot)$ by

$$a(u^h, v^h) = (Au, v).$$

Then

$$(A_i^{(k)}\phi_i^{(k)}, \phi_i^{(k)}) = (A\phi_i^{(k)}, \phi_i^{(k)}) = a(\phi_i^{(k)}, \phi_i^{(k)}) = (\alpha_i^{(k)}\phi_i^{(k)}, \phi_i^{(k)})_{L_2(\Omega)},$$

where $\alpha_i^{(k)} = \dfrac{a(\phi_i^{(k)}, \phi_i^{(k)})}{(\phi_i^{(k)}, \phi_i^{(k)})_{L_2(\Omega)}}$. We have the following equalities:

$$A_i^{(k)}\phi_i^{(k)} = \frac{a(\phi_i^{(k)}, \phi_i^{(k)})}{(\phi_i^{(k)}, \phi_i^{(k)})_{L_2(\Omega)}}\phi_i^{(k)},$$
$$(A_i^{(k)})^{-1}\phi_i^{(k)} = \frac{(\phi_i^{(k)}, \phi_i^{(k)})_{L_2(\Omega)}}{a(\phi_i^{(k)}, \phi_i^{(k)})}\phi_i^{(k)}.$$

Hence we have the following relations for the preconditioner B

$$B^{-1}u^h = \sum_{k=0}^{J}\sum_{i=0}^{n_k}(A_i^{(k)})^{-1}Q_i^{(k)}u^h = \sum_{k=0}^{J}\sum_{i=0}^{n_k}\frac{(u^h, \phi_i^{(k)})_{L_2(\Omega)}}{a(\phi_i^{(k)}, \phi_i^{(k)})}\phi_i^{(k)}.$$

Remark 9.4 If $a(\phi^h, \phi^h) = O(1)$, it is possible to replace the above preconditioner by the original BPX preconditioner

$$B_{BPX}^{-1}u^h = \sum_{k=0}^{J}\sum_{i=0}^{n_k}(u^h, \phi_i^{(k)})_{L_2(\Omega)}\phi_i^{(k)}.$$

10 The Fictitious Space Method

The ASM decomposes an original problem into subproblems. But in some cases (subproblems with unstructured grids) a construction of effective preconditioners is still a difficult problem. To solve an original problem we will use the combination of ASM and Fictitious Space Method (FSM). FSM is the generalization of Fictitious Domain Method. Some references on the Fictitious Domain Methods are given in the introduction.

The following abstract theorem is the basis for the FSM.

Theorem 10.1 *Let H_0 and H be two Hilbert spaces with the scalar products $(\cdot,\cdot)_{H_0}$, and $(\cdot,\cdot)_H$, respectively. Let $A : H_0 \to H_0$ and $B : H \to H$ be some self adjoint positive definite operators, i.e., $A^* = A > 0$ and $B^* = B > 0$. Assume that there exists an operator $R : H \to H_0$ such that*

$$(ARv, Rv)_{H_0} \le C_R(Bv, v)_H \qquad \forall v \in H,$$

and an operator $T : H_0 \to H$ *such that*

$$RTu_0 = u_0 \qquad \forall u_0 \in H_0,$$

and

$$C_T(BTu_0, Tu_0)_H \le (Au_0, u_0)_{H_0} \qquad \forall u_0 \in H_0.$$

Set $C^{-1} = RB^{-1}R^*$*, where* $R^* : H \to H_0$ *and* $(R^*u_0, v)_H = (u_0, Rv)_{H_0}$*. Then the spectral equivalence inequalities*

$$C_T(A^{-1}u_0, u_0)_{H_0} \le (C^{-1}u_0, u_0) \le C_R(A^{-1}u_0, u_0) \qquad \forall u_0 \in H$$

are valid.

The proof of this theorem uses the following result.

Lemma 10.2 *Let* $A = A^* > 0$ *in a Hilbert space with scalar product* $(\cdot, \cdot)$*. Then the identity*

$$(A^{-1}u, u)^{1/2} = \sup_{v \in H} \frac{(u, v)}{(Av, v)^{1/2}}$$

holds.

Proof (Lemma 10.2). By the Cauchy–Bunjakowski inequality, one obtains

$$(u, v) = (A^{-1/2}u, A^{1/2}v) \overset{\text{C.B.}}{\le} \|A^{-1/2}u\| \|A^{1/2}v\| = (A^{-1}u, u)^{1/2}(Av, v)^{1/2}.$$

With $v = A^{-1}u$, one can conclude that

$$(A^{-1}u, u)^{1/2} = \sup_{v \in H} \frac{(u, v)}{(Av, v)^{1/2}},$$

which proves the lemma. □

Proof (Theorem 10.1). In order to prove the lower estimate, we use the above assumptions about R and T and Lemma 10.2. This gives

$$\begin{aligned}
(RB^{-1}R^*u_0, u_0)^{1/2})_{H_0} &= (B^{-1}R^*u_0, R^*u_0)_H = \sup_{v \in H} \frac{(R^*u_0, v)_H}{(Bv, v)_H^{1/2}} \\
&\ge \sup_{v_0 \in H_0} \frac{(R^*u_0, Tv_0)_H}{(BTv_0, Tv_0)_H^{1/2}} \ge \sqrt{C_T} \sup_{v_0 \in H_0} \frac{(R^*u_0, Tv_0)_H}{(Av_0, v_0)^{1/2}} \\
&= \sqrt{C_T} \sup_{v_0 \in H_0} \frac{(u_0, v_0)_{H_0}}{(Av_0, v_0)^{1/2}} = \sqrt{C_T}(A^{-1}u_0, u_0)^{1/2}.
\end{aligned}$$

For the upper estimate, we have

$$\begin{aligned}
(RB^{-1}R^*u_0, u_0)^{1/2})_{H_0} &= \sup_{v \in H} \frac{(u_0, Rv)_{H_0}}{(Bv, v)_H^{1/2}} = \sup_{v \in H} \frac{(A^{-1/2}u_0, A^{1/2}Rv)_{H_0}}{(Bv, v)_H^{1/2}} \\
&\overset{\text{C.B.}}{\le} (A^{-1}u_0, u_0)_{H_0}^{1/2} \sup_{v \in H} \frac{(ARu_0, Rv)^{1/2}}{(Bv, v)^{1/2}} \\
&\le \sqrt{C_R}(A^{-1}u_0, u_0)^{1/2}.
\end{aligned}$$

This completes the proof of Theorem 10.1. □

11 Application to the Fictitious Domain Method

In this section we show how the general framework of FSM can be used for the analysis of the classical Fictitious Domain Methods. However, we have to distinguish between different cases of the boundary conditions.

11.1 Neumann Boundary Condition

The simplest case is the case of Neumann boundary conditions. Let us consider the following model problem:

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma, \end{cases}$$

where Ω is not regular (not polygonal) and Γ is its boundary. Let Π be a domain of much simpler form which includes the domain Ω. A possible candidate for Π is a cube.

Let $H_0 = H^1(\Omega)$ and $H = H_0^1(\Pi)$. Let A and B be the differential operators according to the domain Ω and Π, i.e.,

$$A \longleftrightarrow -\Delta_\Omega + I, \qquad B \longleftrightarrow -\Delta_\Pi.$$

Let $R : H_0^1(\Pi) \to H^1(\Omega)$ be a restriction operator. In this case, we define it by $R = I_\Omega$. Then we have

$$(Ru, Ru)_{H^1(\Omega)} \leq C_R(\nabla u, \nabla u)_{L_2(\Pi)}.$$

Let $T : H^1(\Omega) \to H_0^1(\Pi)$ be an extension operator. For any $u \in H^1(\Omega)$, we have

$$\|u\|_{H^1(\Omega)} \geq C_1 \|\phi\|_{H^{1/2}(\Gamma)} \geq C_2 \|Tu\|_{H^1(\Pi)}$$

with

$$RTu_0 = u_0 \qquad \forall u_0 \in H^1(\Omega).$$

Hence, we obtain the preconditioner for the domain Ω by setting

$$C^{-1} = RB^{-1}R^*.$$

In matrix notation, we choose

$$C^{-1} = \begin{bmatrix} I & 0 \end{bmatrix} (-\Delta_\Pi^{-1}) \begin{bmatrix} I \\ 0 \end{bmatrix} \quad \text{and} \quad Ru = u_{\bar{\Omega}},$$

where

$$R = \begin{bmatrix} I & 0 \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} u_{\bar{\Omega}} \\ u_{\Pi \setminus \Omega} \end{bmatrix}$$

and I is an identity block.

11.2 Dirichlet Boundary Condition (1-D Case)

The situation for Dirichlet boundary conditions is much more difficult. In order to understand the difficulties, we start with the one-dimensional case and investigate the boundary value problem

$$\begin{cases} -\dfrac{d^2u}{dx^2} = f \text{ in } (a,b) \subset (0,1), \\ \qquad u(a) = u(b) = 0. \end{cases}$$

Let $H_0 = H_0^1(a,b)$ and $H = H_0^1(0,1)$ with $\Pi = (0,1)$ and $\Omega = (a,b)$. Let $A = -\Delta_\Omega$ and $B = -\Delta_\Pi$. In order to extend u from Ω to u on Π, we define an extension operator $T : H_0^1(a,b) \to H_0^1(0,1)$ by

Note 32: Or: extend u from Ω to Π

$$Tu = \begin{cases} u(x), & x \in (a,b), \\ 0, & x \in \Pi \setminus (a,b). \end{cases}$$

Then the relation $(Tu_0, Tu_0)_{H^1(\Pi)} = (u_0, u_0)_{H^1(\Omega)}$ implies $C_T = 1$. Next, we consider the restriction operator $R : H^1_{0,h}(\Pi) \to H^1_{0,h}(\Omega)$. There are many ways to define R. Here we investigate two different definitions of R which we will compare.

(1) The first definition is as follows:

$$Ru^h = \begin{cases} u^h(x), & x_i \in (a,b), \\ 0, & x_i = a \text{ or } b. \end{cases}$$

Note that $\|R\| \to \infty$ as $h \to 0$ which implies $C_T \to \infty$. This is not a good choice for a possible restriction operator.

(2) Thus, we introduce another restriction operator. Let $I_\Omega : H_0^1(\Omega) \to H^1(\Omega)$ be the natural restriction defined as follows:

$$(I_\Omega u)(x) = u(x), x \in \Omega \quad \forall u \in H_0^1(\Omega),$$

and $I_\Gamma : H_0^1(\Pi) \to R^2$ be the trace operator defined by

$$I_\Gamma u = \begin{bmatrix} u(a) \\ u(b) \end{bmatrix} \qquad \forall u \in H_0^1(\Pi).$$

Let $t : R^2 \to H^1(\Omega)$ be the extension operator defined by

$$t\left(\begin{bmatrix} u(a) \\ u(b) \end{bmatrix}\right) = u(a) + \frac{u(b) - u(a)}{b-a}(x-a).$$

Now we define the restriction operator $R : H_0^1(\Pi) \to H_0^1(\Omega)$ by

$$R = I_\Omega - tI_\Gamma. \tag{11.1}$$

Clearly, we have the estimates

$$|u(a)| \leq C\|u\|_{H^1(\Pi)}, \qquad |u(b)| \leq C\|u\|_{H^1(\Pi)},$$

and

$$|tI_\Gamma u|^2_{H^1} = \int_a^b \frac{(u(b)-u(a))^2}{(b-a)^2} dx \leq C\|u\|^2_{H^1(\Pi)}.$$

This implies

$$\|Ru\|_{H^1(\Omega)} \leq \|I_\Omega u\|_{H^1(\Omega)} + \|tI_\Gamma u\|_{H^1(\Omega)} \leq C_R\|u\|_{H^1(\Pi)}$$

since $\|I_\Omega u\|_{H^1(\Omega)} \leq \|I_\Omega u\|_{H^1(\Pi)}$. Note that the constant C_R is independent of h in this case (FEM). It is easy to see that $RTu_0 = u_0 - 0 = u_0$ for all $u_0 \in H^1_0(\Omega)$.

Summarizing, only the definition of R in (11.1) leads to a restriction operator with constant c_R bounded independently of the meshsize h.

11.3 Dirichlet Boundary Condition (2-D Case)

Next, we consider the two-dimensional case. Let $H_0 = H^1_0(\Omega)$, $H = H^1_0(\Pi)$, $A = -\Delta_\Omega$ and $B = -\Delta_\Pi$. The operator T is defined as in the one-dimensional case, i.e. let $T : H^1_0(\Omega) \to H^1_0(\Pi)$ with

$$Tu = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \Pi \setminus \Omega. \end{cases}$$

This gives $C_T = 1$. For the definition of R, we have generalize the definition in (11.1) and introduce $R = I_\Omega - tI_\Gamma$, where t is the extension operator from Section 7.2. Then we obtain a constant C_R (independent of h).

Note 33: generalized or to generalize?

11.4 Mixed Boundary Condition (2-D case)

In the case of mixed boundary conditions, the ideas for the Dirichlet and the Neumann case have to be combined. Let $\breve{H}^1(\Omega) = \{u \in H^1(\Omega) | u(x) = 0, x \in \Gamma_D\}$. Let $H = H^1_0(\Pi)$, $A = -\Delta_\Omega$ and $B = -\Delta_\Pi$. We consider the subdomains G_N and G_D in the vicinity of Γ_N and Γ_D, respectively, such that

$$\overline{\Pi \setminus \Omega} = \overline{G}_N \cup \overline{G}_D$$

and

$$\partial G_D \cap \Gamma_N = \emptyset, \quad \partial G_N \cap \mathring{\Gamma}_D = \emptyset.$$

Let $T_{ND}u_0 = T_N T_D u_0$ where the operator $T_D : \breve{H}^1(\Omega) \to H^1(\bar{\Omega} \cup G_D)$ for the Dirichlet data is defined by

$$T_D u_0 = \begin{cases} u_0(x), & x \in \Omega, \\ 0, & x \in G_D. \end{cases}$$

Next, by the trace theorem, there exists an operator $T_N : H^1(\bar{\Omega} \cup G_D) \to H^1(\Pi)$. Now, we define a restriction operator R by $R = I_\Omega - t_\Gamma t_N \cdot I_D$ where $I_\Omega : H_0^1(\Pi) \to H^1(\Omega)$ and $I_D : H_0^1(\Pi) \to H^{1/2}(\Gamma_D)$. We define $t_N : H^{1/2}(\Gamma_D) \to H^{1/2}(\Gamma)$ by

$$(t_N\phi)(-s) = (1 - \frac{s}{D})\phi(s) \quad \text{for } \phi(s) \in H^{1/2}(\Gamma_D).$$

Here $(1 - s/D)$ is a linear cut-off function. We note that D is independent of h. For t_N, the estimate

$$\|t_N\phi\|_{H^{1/2}(\Gamma)} \le C_1 \|\phi\|_{H^{1/2}(\Gamma_D)}.$$

can easily be proved. Let $t_\Gamma : H^{1/2}(\Gamma) \to H^1(\Omega)$ be the extension operator of Section 7.2. Then we obtain the estimates

$$\|Ru\| \le \|I_\Omega\| + \|t_\Gamma\| \cdot \|t_N\| \cdot \|I_D u\| \le C_R \|u\|_{H^1(\Pi)},$$

where C_R is independent of h in the FEM case.

11.5 Unstructured and Nonuniform Grid (2-D Case)

In the previous sections, we have investigated the case of a structured grid. Here we consider the case of an unstructured and nonuniform grid for Ω. In other words, we can design a preconditioner for the differential operator on Ω from that on Π. In the case when Ω is not polygonal, though, we want to design a preconditioner from the uniform grid differential operator on Π. Let $Q^{h_\sharp}$ denote the uniform grid on Π and $h_\sharp$ be the mesh size of $Q^{h_\sharp}$ satisfying

$$h_\sharp < \frac{1}{\sqrt{2}} r_{\min} \quad \text{where} \quad r_{\min} = \min_{z_l \in \Omega^h} r_l.$$

Here r_l is the radius of the largest ball $B(z_l, r_l)$ inscribed in the union of all elements of the triangulation Ω^h sharing the vertex z_l.

Let $H_0 = H_h(\Pi^h)$ and $H = H_h(Q^{h_\sharp})$. Let $A = -\Delta_{\Pi^h}$ and $B \approx -\Delta_{Q^{h_\sharp}}$ be defined as in the previous sections. Now, we introduce a restriction operator

$$R_Q : H_h(Q^{h_\sharp}) \to H_h(\Pi^h), \tag{11.2}$$

i.e. for any $U^{h_\sharp} \in H_h(Q^{h_\sharp})$ the values $u^h \in H_h(\Pi^h)$ have to be defined. Let z_l denote the nodal point of Π^h and Z_{ij} is the node of some Q_{ij}. We set $u^h(z_l) = U^{h_\sharp}(Z_{i,j})$, i.e., $RU^{h_\sharp} = u^h$ is a simple restriction. Next, we define the extension operator $T : H_h(\Pi^h) \to H_h(Q^{h_\sharp})$ by the following way:

$$\begin{cases} U^{h_\sharp}(z_{ij}) = u^h(z_l), & \text{if } z_l \text{ belongs to some } Q_{ij}, \\ U^{h_\sharp}(z_{ij}) = \frac{1}{3}(u^h(z_1) + u^h(z_2) + u^h(z_3)), & \text{otherwise .} \end{cases}$$

With the condition on the mesh size there are only two cases. That is, there is a one-to-one correspondence between Π^h and some subset $\tilde{Q}^{h_\sharp}$ of $Q^{h_\sharp}$. Then we can see that

$$RTu^h = u^h \qquad \forall u^h \in H_h(\Pi^h).$$

Lemma 11.1 *Let us assume that* $c_1 r_{\min} \leq h_\sharp$*, i.e.,* $h_\sharp$ *is of order* h*. Then, there exist two constants* C_R^Q *and* C_T^Q *(independent of* h*) such that*

$$\|R_Q U^{h_\sharp}\|_{H^1(\Pi)} \leq C_R^Q \|U^{h_\sharp}\|_{H^1(\Pi)} \quad \textit{and} \quad \|T_Q u^h\|_{H^1(\Pi)} \leq C_T^Q \|u^h\|_{H^1(\Pi)}.$$

Proof. Using $u^h = RU^{h_\sharp}$, we can estimate

$$\begin{aligned}
&\|u^h\|_{H^1(\Pi)} \\
&\approx \sum_{\tau_i \subset \Pi^h} (h^2\{(u^h(z_{i_1}))^2 + (u^h(z_{i_2}))^2 + (u^h(z_{i_3}))^2\} \\
&\qquad + (u^h(z_{i_1}) - u^h(z_{i_2}))^2 + (u^h(z_{i_2}) - u^h(z_{i_3}))^2 + (u^h(z_{i_3}) - u^h(z_{i_1}))^2) \\
&= \sum_{\tau_i \subset \Pi^h} h^2((U^{h_\sharp}_{i_1,j_1})^2 + \cdots + (U^{h_\sharp}_{i_k,j_k})^2) \\
&\qquad + \sum_{\tau_i \subset \Pi^h} ((U^{h_\sharp}_{i_1,j_1} - U^{h_\sharp}_{i_2,j_2})^2 + (U^{h_\sharp}_{i_2,j_2} - U^{h_\sharp}_{i_3,j_3})^2 + (U^{h_\sharp}_{i_3,j_3} - U^{h_\sharp}_{i_1,j_1})^2).
\end{aligned}$$

Clearly,

$$\sum_{\tau_i \subset \Pi^h} h^2((U^{h_\sharp}_{i_1,j_1})^2 + \cdots + (U^{h_\sharp}_{i_k,j_k})^2) \leq \|U^{h_\sharp}\|_{L_2,h}(Q^{h_\sharp}).$$

Next,

$$\begin{aligned}
(U^{h_\sharp}_{i_1,j_1} - U^{h_\sharp}_{i_2,j_2})^2 &\leq \text{some differences of neighbors} \\
&\leq (U^{h_\sharp}_{i_1,j_1} - U^{h_\sharp}_{i_2',j_2'})^2 + \cdots + (U^{h_\sharp}_{i_k',j_k'} - U^{h_\sharp}_{i_2,j_2})^2.
\end{aligned}$$

Thus, there exists a constant C_1 such that

$$\sum_{\tau_i \subset \Pi^h} ((U^{h_\sharp}_{i_1,j_1} - U^{h_\sharp}_{i_2,j_2})^2 + (U^{h_\sharp}_{i_2,j_2} - U^{h_\sharp}_{i_3,j_3})^2 + (U^{h_\sharp}_{i_3,j_3} - U^{h_\sharp}_{i_1,j_1})^2) \leq C_1 |U^{h_\sharp}|^2_{H^1(Q^{h_\sharp})}.$$

This completes the proof of the existence of C_R. The proof of the existence of C_T is the same as the case of C_R. □

Let

$$C^{-1}_{\text{FSM},u} = RR_Q(-\Delta_{Q^{h_\sharp}})^{-1} R_Q^\top R^\top \tag{11.3}$$

be the FSM preconditioner for unstructured grids. Using Theorem 10.1 twice and Lemma 11.1, we have proved the following final result about the Fictitious space method.

Theorem 11.2 *Let* $C_{\text{FSM},u}$ *be defined via* (11.3)*. Then the spectral equivalence relation* $\triangle\Omega_h \sim C_{\text{FSM},u}$ *holds.*

12 Fictitious Space Method and Multilevel ASM

In the previous section, the FSM-preconditioner $C_{\text{FSM},u}$ (11.3) has been developed. For this preconditioner, we have to solve a potential problem on a much simpler geometry. In this section, we consider the following mixed boundary value problem:

$$\begin{aligned} -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} + a_0(x)u = f(x), &\qquad x \in \Omega, \\ u(x) = 0, &\qquad x \in \Gamma_D, \\ \frac{\partial u}{\partial n} + \sigma(x)u = 0, &\qquad x \in \Gamma_N, \end{aligned}$$

where the diffusion matrix $(a_{ij})_{i,j=1}^{2}$ is symmetric and positive definite, and the coefficients a_0 and σ are uniformly positive. Then, we introduce the corresponding bilinear form

$$a(u,v) = \int_\Omega \left(\sum_{i,j=1}^{2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0(x)uv \right) dx + \int_{\Gamma_N} \sigma(x)uv\, ds,$$

which satisfies the relations

$$a(u,v) = a(v,u) \quad \text{and} \quad a(u,u) \approx \|u\|^2_{H^1_\Omega}. \tag{12.1}$$

A flexible domain decomposition method was suggested in [4]. This method is not optimal with respect to convergence rate, but it is simple to implement and can be very effective for parabolic problems. To suggest an optimal algorithm we consider the following approach. We assume that the triangulation $\overline{\Omega}^h = \bigcup_{i=1}^{M} \overline{\tau}_i$ is quasi-uniform and shape regular with $\partial\Omega^h = \Gamma_D^h \cup \Gamma_N^h$ and $\Gamma_D^h \subset \overline{\Omega}, \Gamma_N^h \subset \overline{(R^2 \setminus \Omega)}$. Also suppose that $h_\sharp \leq r_{\min}/2\sqrt{2}$. We define an auxiliary mesh Q^h by the minimum collection of Q_{ij} enclosing Ω^h. Let $\partial Q^h = S^h$ with $S^h = S_D^h \cup S_N^h$ such that, if $\overline{Q}_{ij} \cap \Gamma^h \neq \emptyset$, then

$$S^h \cap \overline{Q}_{ij} \in S_D^h \quad \text{and} \quad S_N^h = S^h \setminus S_D^h.$$

The operator induced by (12.1) is defined as

$$(Au,v) = a(u^h, v^h).$$

Moreover, let B be an operator which satisfies

$$(BU,U) \approx \|U^h\|^2_{H^1(Q^h)} \quad \forall U^h \in H_h(Q^h)$$

and

$$R_Q : H_h(Q^h) \to H_h(\Omega^h)$$

be defined via (11.2).

Theorem 12.1 *There exist two constants C_1 and C_2, independent of h, such that*

$$C_1\,(A^{-1}u,v) \le (RB^{-1}R^\top u,v) \le C_2\,(A^{-1}u,u) \quad \forall u.$$

Proof. The proof is similar to the proof of Theorem 11.2. □

Remark 12.2 The condition $h_\sharp \le r_{\min}/2\sqrt{2}$ instead of $h_\sharp < r_{\min}/\sqrt{2}$ is needed in the mixed boundary case.

First we consider $\Gamma_D = \emptyset$, i.e. the case of pure Neumann boundary conditions.

Let us assume that $h_\sharp = l \cdot 2^{-J}$, $h_k = l \cdot 2^{-k}$, $k = 0, 1, \ldots, J$, and $h_J = h_\sharp$, and that we have a sequence of triangulations

$$\Pi_0^h, \Pi_1^h, \ldots, \Pi_J^h$$

and spaces

$$W_0^h \subset W_1^h \subset \cdots \subset W_J^h = \breve{H}_h(Q^h),$$

where $W_k^h = \{u_k^h = \sum_i \alpha_i^{(k)}\phi_i^{(k)}\}$ and $\phi_i^{(k)}$ is a nodal basis. Let $S^h = S_N^h$. Now, we introduce the preconditioner in Π via

$$B_N^{-1}U^h = \sum_{k=0}^{J} \sum_{\{\mathrm{supp}\phi_i^{(k)} \cap Q^h \ne \phi\}} (\tilde{U}^h, \phi_i^{(k)})_{L_2(Q^h)}\tilde{\phi}_i^{(k)}, \tag{12.2}$$

where $\tilde{U}^h = U^h(Z_{i,j})$ for $Z_{i,j} \in Q^h$ and $\tilde{U}^h = 0$ otherwise.

Theorem 12.3 *Let B_N be defined via* (12.2)*. Then there exist two constants C_1 and C_2, which are independent of h, such that*

$$C_1\,\|U^h\|^2_{H^1(Q^h)} \le (B_N U, U) \le C_2\,\|U^h\|^2_{H^1(Q^h)} \quad \forall U^h \in H_h(Q^h).$$

Proof. Note that

$$B_\Pi^{-1}(U_\Pi^h) = \sum_{k=0}^{J}(U_\Pi^h, \Phi_i^{(k)})_{L_2(\Pi)}\Phi_i^{(k)} \qquad \text{(BPX in } \Pi),$$

$$R_N U_\Pi^h = \begin{cases} U_\Pi^h(Z_{i,j}), & Z_{i,j} \in Q^h, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$R_N = \begin{bmatrix} I & 0 \end{bmatrix}.$$

Then we have $R_N B_N^{-1} R_N^\top = B_N^{-1}$ which proves the theorem. □

Next, we consider the case of Dirichlet boundary conditions ($S^h = S_D^h$), i.e. $\Gamma_N = \emptyset$. Here, we define

$$B^{-1}U^h = \sum_{k=0}^{J} \sum_{\mathrm{supp}\Phi_i^{(k)} \subset Q^h} (U_\Pi^h, \Phi_i^{(k)})_{L_2(\Pi)}\Phi_i^{(k)}.$$

Theorem 12.4 *There exist two constants C_1 and C_2, which are independent of h, such that*

$$C_1\|U^h\|^2_{H^1(Q^h)} \le (B_D U, U) \le C_2\|U^h\|^2_{H^1(Q^h)} \quad \forall U \in H_h(Q^h).$$

Proof. The proof is moved to the end of this section. Let us assume temporarily that the theorem was proved already. □

At last we consider the case of mixed boundary conditions ($S^h_D \neq \emptyset, S^h_N \neq \emptyset$) and define

$$B_M^{-1}U = \sum_{k=0}^{J}\sum (U^h, \Phi_i^{(k)})_{L_2(\Pi)}\Phi_i^{(k)}$$

as the BPX preconditioner. Note that the second summation is taken on the set supp$(\Phi_i^{(k)} \cap Q^h) \neq \phi$ and supp$(\Phi_i^{(k)} \cap S^h_D) = \phi$.

Note 34: ϕ or $\emptyset$?

Theorem 12.5 *There exist positive constants C_1 and C_2 such that*

$$C_1\,\|U^h\|^2_{H^1(Q^h)} \le (B_M U, U) \le C_2\,\|U^h\|^2_{H^1(Q^h)}\ \forall\, U \in H_h(Q^h).$$

Proof. Note that $\Pi^h\backslash Q^h = \bar{G}^h_D \cup \bar{G}^h_N$ and $G^h_D \cap G^h_N = \emptyset$. We have that $\partial G^h_D \cap S^h = S^h_D$ and $\bar{G}^h = \bar{Q}^h \cup \bar{G}^h_N$. Now we define

$$\dot{H}_h(G^h) = \{u^h | u^h(x) = 0, x \in \partial G^h\}.$$

Then we have by the previous case

$$B^{-1}_{D,G}U^h_G = \sum_{k=0}^{J}\sum_{\text{supp}\Phi_i^{(k)} \subset G^h}(U^h_G, \Phi_i^{(k)})_{L_2(G)}\Phi_i^{(k)}$$

and

$$C_1\|G^h_G\|^2_{H^1(G)} \le (B_{D,G}U_G, U_G) \le C_2\|U^h_G\|^2_{H^1(G)}.$$

Furthermore, let us define $R_{N,G} : \dot{H}_h(G^h) \to \breve{H}_h(Q^h)$ by

$$R_{N,G}U^h_G(Z_{i,j}) = \begin{cases} U^h_G(Z_{i,j}), & Z_{i,j} \in Q^h, \\ 0, & \text{otherwise.} \end{cases}$$

Then we finally have $R_{N,G}B^{-1}_{D,G}R^{-1}_{N,G} = B^{-1}_M$ with $R_{N,G} = \begin{bmatrix} I & 0 \end{bmatrix}$. □

Now we are in a position to show Theorem 12.4 for the case of Dirichlet boundary conditions. We define $\dot{W}_k = W_k \cap \dot{H}_h(Q^h)$. Then the proof of the theorem is completed if the following conditions are satisfied:

(a) For all $u^h \in \dot{W}_J$, there exists $u_i^{(k)} = \alpha_i^{(k)}\Phi_i^{(k)}$ such that

$$\sum_{k=0}^{J}\sum_{\text{supp}(u_i^{(k)}) \subset Q^h} u_i^{(k)} = u^h \quad \text{and} \quad \alpha \sum_{\text{supp}(u_i^{(k)}) \subset Q^h} \|u_i^{(k)}\|^2_{H^1(Q^h)} \le \|u^h\|^2_{H^1(\Omega)},$$

and,

(b) for all $u^h \in \dot{W}_J$, the inequality

$$\|u^h\|^2_{H^1(Q^h)} \le \beta \inf \sum_{\text{supp}(u_i^{(k)}) \subset Q^h} \|u_i^{(k)}\|^2_{H^1(Q^h)}$$

is valid, where the infimum is taken over all decompositions satisfying

$$\sum_{k=0}^{J} \sum_{\text{supp}(u_i^{(k)}) \subset Q^h} u_i^{(k)} = u^h.$$

The positive constants are supposed to be independent of h.

Now, in order to prove the above conditions (a) and (b), we need three lemmas. The proof of each lemma is easy, so omitted. The first and second lemma imply the condition (b) and the last lemma with the BPX preconditioner in Π implies the condition (a). Finally, we will now state the three lemmas.

Lemma 12.6 *There exists C, independent of h, such that*

$$(\nabla v^h, \nabla w^h)_{L_2(\tau_i)} \le C(1/\sqrt{2})^{l-k} |v|_{H^1(\tau_i)} 2^l \|w\|_{L_2(\tau_i)}$$

for all triangles τ_i of the triangulation $\Pi_k^h \cap SuppW_k$, $v^h \in \dot{W}_k$, $w^h \in \dot{W}_l$ $(l > k)$. **Note 35:** Supp or supp?

Lemma 12.7 *For all $u^h = u_0^h + \sum_{k=1}^{J} u_k^h$, $u_k^h \in \dot{W}_k$, we have*

$$|u^h|^2 \le C(|u_0^h|^2_{H^1(Q^k)} + \sum_{k=1}^{J} 4^k \|u_k^h\|^2_{L_2(Q^h)}).$$

Lemma 12.8 *For given $u^h \in \dot{W}_J$, we define $\tilde{u}^h(x) = u^h(x)$ if $x \in Q^h$, otherwise $\tilde{u}^h = 0$. Then, for a given decomposition*

$$\tilde{u}^h = \tilde{u}_0 + \sum_{k=1}^{J} \tilde{u}_k, \quad \tilde{u}_k \in W_k,$$

there exists a decomposition

$$u^h = u_0 + \sum_{k=1}^{J} u_k, \quad u_k \in \dot{W}_k$$

such that

$$4^k \|u_k\|^2_{L_2(Q^h)} \le C(|\tilde{u}_0| + \sum_{k=1}^{J} \|\tilde{u}_k\|^2_{L_2(\Pi)})$$

for some constant C, independent of h.

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