3.2.2. The Hellinger-Reisner Principle

- **Idea:** Find displacements $\overset{\text{dual}}{u}$ and stresses $\overset{\text{primal}}{\sigma}$:

(13)
$$\boxed{\begin{aligned} &D^{-1}\sigma - \varepsilon(u) = 0 \\ &\operatorname{div}\sigma = -f \\ &+ BC:\ u = \bar{u}\ (:= 0)\ \text{on } \Gamma_u = \Gamma_D \ \text{and} \\ &\qquad \sigma_n := \sigma\cdot n = t \ \text{on } \Gamma_t \end{aligned}}$$

Remark: Compare with Example 1.2:
→ Dirichlet Problem for the Poisson equation

$$\boxed{\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \Gamma \end{aligned}} \longrightarrow \boxed{\begin{aligned} &\sigma - \nabla u = 0 \text{ in } \Omega \\ &\operatorname{div}\sigma = -f \text{ in } \Omega \\ &+ BC:\ u = 0 \text{ on } \Gamma_u = \Gamma \end{aligned}}$$

- **Mixed Formulation in $L_2(\Omega) \times H_0^1(\Omega)$:**
= primal mixed formulation (PMF)

  - Idea: ① Weak formulation of $D^{-1}\sigma - \varepsilon(u) = 0$

    ② Weak formulation of $\operatorname{div}\sigma = -f$
    + integration by parts

  - Let us define the spaces:

$$X = \{\sigma = [\sigma_{ij}] : \sigma_{ij} = \sigma_{ji} \in L_2(\Omega)\} = \mathbb{L}_2(\Omega) = \mathbb{L}_2^{sym}(\Omega)$$

$$\Lambda = \{v = [v_i] : v_i \in H^1(\Omega),\ v = 0 \text{ on } \Gamma_u\}$$
$$= H_0^1(\Omega) \times H^1_{0,\Gamma_u}(\Omega) = \bar{V}_0 .$$