■ KORN's Inequalities:

- For the space $V_0 = [\overset{\circ}{H}{}^1(\Omega)]^3$, i.e. 1st BVP:

Lemma 3.3: (1st Korn's inequality)

$$(6)\quad \boxed{\begin{array}{l} \int_\Omega \varepsilon_{ij}(v)\,\varepsilon_{ij}(v)\,dx \geq \frac{1}{2}\int_\Omega v_{i,j}\,v_{i,j}\,dx \quad \forall v \in [\overset{\circ}{H}{}^1(\Omega)]^3 \\ \qquad \| \qquad\qquad\qquad\qquad\qquad \| \\ \|\varepsilon(v)\|^2_{0,\Omega} \qquad \geq \; c_K^2\, |v|^2_{1,\Omega} \qquad -\|- \end{array}}$$

Proof:

It is obviously sufficient, to prove (6) for smooth vector fields $v \in [\overset{\circ}{C}{}^\infty(\Omega)]^3$ (closure principle!).
For $v \in [\overset{\circ}{C}{}^\infty(\Omega)]^3$, we obviously have

$$\varepsilon_{ij,j}(v) = \frac{1}{2}(v_{i,j} + v_{j,i})_{,j} =$$

$$= \frac{1}{2}(v_{i,jj} + v_{j,ij}) = \frac{1}{2}(\Delta v_i + (\operatorname{div}(v))_{,i}).$$

Using this identity, we obtain by integration by parts

$$\int_\Omega \varepsilon_{ij}(v)\,\varepsilon_{ij}(v)\,dx = \frac{1}{2}\int_\Omega \overset{\varepsilon_{ji}(v)}{\overset{\shortparallel}{\varepsilon_{ij}(v)}}\,(v_{i,j} + v_{j,i})\,dx$$

$$= \int_\Omega \varepsilon_{ij}(v)\, v_{i,j}\,dx \underset{\text{by parts}}{\overset{\text{integr.}}{=}} -\int_\Omega \varepsilon_{ij,j}(v)\cdot v_i\,dx + \overset{v|_\Gamma = 0}{0} =$$

$$= -\frac{1}{2}\int_\Omega [\overset{v_{i,jj}}{\Delta v_i} + (\operatorname{div}(v))_{,i}]\, v_i\,dx$$

$$\underset{\text{integ. by parts}}{\overset{v|_\Gamma=0}{=}} \frac{1}{2}\int_\Omega [v_{i,j}\,v_{i,j} + (\operatorname{div}(v))^2]\,dx$$

$$\geq \frac{1}{2}\int_\Omega v_{i,j}\cdot v_{i,j}\,dx$$

q.e.d.